	GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503. (AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI) DEPARTMENT OF MATHEMATICS STUDY E - MATERIAL CLASS : II B.SC (MATHEMATICS) SEM: I SUBJECT : CALCULUS

MSU/ 2017-18 / UG-Colleges /Part-III (B.Sc. Mathematics) / Semester – I / Core-1

CALCULUS (75 Hours)

Unit I : Curvature,Radius of Curvature and Centre of curvature in Cartesian and polar Co-ordinates

Unit II Pedal Equation-Involute and evolute-Asymptotes

Unit III Singular Points(Node,cusp,conjugate points)-Tracing of curves (cartesian only)

Unit IV Double and Triple Integrals - Changing the order of integration - Jacobians and change of variables

Unit V Beta and Gamma functions – Application of Beta and Gamma Functions in evaluation of Double and Triple Integrals, Improper Integrals.

Text Book:

Narayanan S and T.K. Manickavasagam Pillai - Calculus Volume I (2004), Volume II (2004), S. Viswanathan Printer Pvt.Ltd.

Books for Reference :

- Kandasamy P and K. Thilagavathi - Mathematics for B.Sc., Volume II – 2004, S. Chand & Co., New Delhi.
- Apostaol T.M. - Calculus, Vol. I (4^{th} edition) John Wiley and Sons, Inc., Newyork 1991.
- Apostaol T.M. - Calculus, Vol. II (2^{nd} edition) John Wiley and Sons, Inc., New York 1969)
- Stewart, J - Single Variable Calculus (4^{th} edition) Brooks / Cole, Cengage Learning 2010.

Calculus

Unit - I

Curvature Radius of curvature and center of Curvature in - Cartesian and Polar co-ordinates.

Unit - II

Pedel equation - Involute and evolute - Asymptotes.

Unit - III

Singular points (Node, Cusp, Conjugate Points) - Tracing of curves (Cartesian only)

Unit - IV

Double and Trible Integrals - changing the order of Integration - Jacobians and change of variables.

Unit - V

Beta and Gamma functions - Applications of Beta and Gamma funs in evaluation of Double and Trible Integrals, Improper Integrals.

Text Book:

Narayanan . S & T. K. Manickavasagam Pillai - Calculus Volume I & II

Calculus

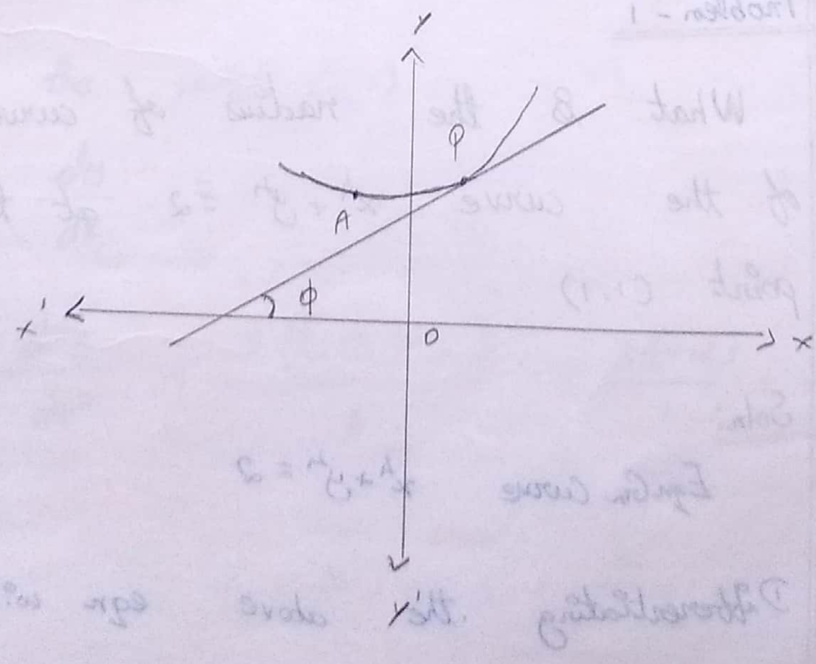
Unit - I

Curvature:

Consider a curve given by the equation $y = f(x)$. Suppose the curve has a definite tangent at each point. Let A be a fixed point on the curve and P be an arbitrary point on the curve. Let S denoted the arclength AP.

Let ϕ be the angle made by the tangent with the x-axis. Then $\frac{d\phi}{ds}$ is called the curvature of the curve at P.

Thus the curvature is the rate of turning of the tangent with respect to the arclength



Definition

The reciprocal of the curvature of a curve at any point is called the radius of curvature at that point and it is denoted by ρ.

$$\rho = \frac{ds}{d\phi}$$

Formula for Radius of Curvature.

i) $$\rho = \frac{(1+\tan^2\phi)^{3/2}}{d^2y/dx^2}$$

ii) $$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

Problem - 1

What is the radius of curvature of the curve $x^4+y^4=2$ at the point $(1,1)$

Soln:

~~Eqn~~ Gn curve $x^4+y^4=2$

Differentiating the above eqn with

respect to x we get

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 4y^3 \frac{dy}{dx} = -4x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\cancel{4}x^3}{\cancel{4}y^3}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x^3}{y^3}$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{y^3(-3x^2) + x^3\left(3y^2 \frac{dy}{dx}\right)}{y^6}$$

$$= \frac{y^{\cancel{2}}\left(3x^3 \frac{dy}{dx} - 3x^2 y\right)}{y^6}$$

$$\frac{d^2y}{dx^2} = \frac{3\left(x^3 \frac{dy}{dx} - x^2 y\right)}{y^4}$$

At the point $(1,1)$

$$\frac{dy}{dx} = -1$$

$$\frac{d^2y}{dx^2} = \frac{3((-1) - 1)}{1} = \frac{3(-2)}{1} = -6$$

The radius of Curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left[1+(-1)^2\right]^{3/2}}{-6}$$

$$= \frac{(2)^{3/2}}{-6}$$

$$= \frac{(8)^{1/2}}{-6} = \frac{2\sqrt{2}}{-6} = \frac{-\sqrt{2}}{3}$$

$$\therefore \boxed{\rho = \frac{-\sqrt{2}}{3}}$$

Pbm - 2

Show that the radius of Curvature at any point of the catenary $y = c \cosh\frac{x}{c}$ is equal to the Length of the portion of the normal intercepted between the curve and the axis of x:

Soln:

Gn Curve : $y = c \cosh\frac{x}{c}$

$$\frac{dy}{dx} = \cancel{c}.\sinh\frac{x}{c}\cdot\frac{1}{\cancel{c}}$$

$$= \sinh\frac{x}{c}$$

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Now $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \left[1+ \sinh^2 \frac{x}{c}\right]^{3/2}$

$$= \left[\cosh^2 \frac{x}{c}\right]^{3/2}$$

$$= \cosh^3 \frac{x}{c}.$$

consider $\frac{d^2y}{dx^2} = \cosh\frac{x}{c} \cdot \frac{1}{c}$

$$= \frac{1}{c}\cosh\frac{x}{c}$$

$\sin hx = \cos hx$

$\cos hx = \sin hx$

$\cos^2 hx = 1+\sin^2 hx$

The Radius of the Curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\cosh^3 x/c}{1/c \cosh x/c}$$

$$= \frac{\cosh^2 \frac{x}{c}}{1/c}$$

$$= c \cdot \cosh^2 \frac{x}{c}$$

$$= \frac{c^2 \cdot \cosh^2 \frac{x}{c}}{c}$$

$$\rho = \frac{y^2}{c}$$

At the point (x,y)

Length of the normal $= y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$

$$= y\left(\cosh^2 \frac{x}{c}\right)^{1/2} = y \cosh \frac{x}{c}$$

$$= \frac{y\left(c \cosh \frac{x}{c}\right)}{c} = \frac{y \cdot y}{c}$$

$$= \frac{y^2}{c}$$

$\therefore$ Radius of the curvature $\Big\}$ = Length of the normal

Pbm - 3

If a curve is defined by the parametric equation $x = f(\theta)$ and $y = \phi(\theta)$, prove that the curvature is

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$ where dashes denote differentiation with respect to w.r.to θ.

Soln:

Gn Curve :- $x = f(\theta)$ $y = \phi(\theta)$.

Let $x' = \frac{dx}{d\theta}$ and $y' = \frac{dy}{d\theta}$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = y' \frac{1}{x'}$

$$= \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{y'}{x'}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{x'y'' - y'x''}{(x')^2} \cdot \frac{1}{x'}$$

$$= \frac{x'y'' - y'x''}{(x')^3}$$

The Curvature $\frac{1}{\rho} = \dfrac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$

$$= \frac{\dfrac{y''x' - y'x''}{(x')^3}}{\left[1+\left(\dfrac{y'}{x'}\right)^2\right]^{3/2}}$$

$$= \frac{y''x' - y'x''}{(x')^3\left[1+\dfrac{y'^2}{x'^2}\right]^{3/2}}$$

$$= \frac{y''x' - y'x''}{(x')^3\,\dfrac{(x'^2+y'^2)^{3/2}}{(x'^2)^{3/2}}}$$

$$= \frac{y''x' - y'x''}{\cancel{(x')^3}\,\dfrac{\left[(x')^2+(y')^2\right]^{3/2}}{\cancel{(x')^3}}}$$

$$= \frac{y''x' - y'x''}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

$$\therefore \boxed{\frac{1}{\rho} = \frac{y''x' - y'x''}{\left[(x')^2 + (y')^2\right]^{3/2}}}$$

Theorem : 1

The Curvature of ~~the~~ a circle of radius r at any point is $\frac{1}{r}$

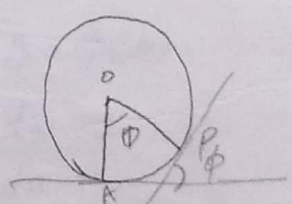
Proof :

Let A be a fixed point on the circle and P be any point on the circle

Let arc AP $= s$. ~~Let the tangent at~~

Let the tangent at P make an angle ϕ with the tangent at A

Then $\angle AOP = \phi$



$\therefore \; s = r\phi$

$$\Rightarrow \frac{ds}{d\phi} = r \quad (\because r \text{ is constant})$$

$$\Rightarrow \frac{d\phi}{ds} = \frac{1}{r}$$

$\therefore$ The curvature of a circle of radius r is $\frac{1}{r}$.

Remark:

For a circle of radius r, the radius of curvature at any point is equal to r.

Pbm - 4

P.T the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ & $y = a(1 - \cos\theta)$ is $4a\cos\theta/2$

Soln:

Gn curve : $x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$

$$x' = \frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$y' = \frac{dy}{d\theta} = a(+\sin\theta) = +a\sin\theta$$

$$x'' = \frac{d^2x}{d\theta^2} = -a\sin\theta$$

$$y'' = \frac{d^2y}{d\theta^2} = a\cos\theta$$

radius of Curvature

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

$$= \frac{\left(a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}{a^2(1+\cos\theta)\cos\theta + a^2\sin^2\theta}$$

$$= \frac{\left(a^2(1+2\cos\theta+\cos^2\theta) + a^2\sin^2\theta\right)^{3/2}}{a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{\left(a^2(1+2\cos\theta) + a^2\right)^{3/2}}{a^2\cos\theta + a^2}$$

$$= \frac{\left[a^2 \cdot (2+2\cos\theta)\right]^{3/2}}{a^2(1+\cos\theta)}$$

$$= \frac{\cancel{a^2}\, a^{\cancel{3}}\left[2(1+\cos\theta)\right]^{3/2}}{\cancel{a^2}(1+\cos\theta)}$$

$$= \frac{a\left[2\left(2\cos^2\theta/2\right)\right]^{3/2}}{2\cos^2\theta/2}$$

$$= \frac{a\left[4\cos^2\theta/2\right]^{3/2}}{2\cos^2\theta/2}$$

$$= \frac{a\left[(2\cos\theta/2)^2\right]^{3/2}}{2\cos^2\theta/2}$$

$$= \frac{a(\cancel{8}^{4}\cos^{3}\theta/2)}{2\cos^{3}\theta/2}$$

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$$= 4a\cos\theta/2$$

$$\boxed{\therefore \rho = 4a\cos\theta/2}$$

Pbm - 5

H.W

Find ρ at any point 't' of the curve $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$\rho = at$

$\frac{dx}{dt} = at\cos t$

$\frac{dy}{dt} = at\sin t$

$\frac{dy}{dx} = \tan t$

$\frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t}$

Soln:

Gn curve :-

$$x = a(\cos t + t\sin t)$$

$$y = a(\sin t - t\cos t)$$

$$\frac{dx}{dt} = a\left[\cancel{-\sin t} + t\cos t + \cancel{\sin t}\right]$$

$$\frac{dx}{dt} = at\cos t$$

$$\frac{dy}{dt} = a\left[\cancel{\cos t} - t(-\sin t) - \cancel{\cos t}\right]$$

$$\frac{dy}{dt} = at\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \cancel{at}\sin t \times \frac{1}{\cancel{at}\cos t}$$

$$= \frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(\tan t)$$

$$= \frac{d}{dt}(\tan t)\,\frac{dt}{dx}$$

$$= \sec^2 t \,\frac{1}{at\cos t}$$

$$= \frac{1}{\cos^2 t}\,\frac{1}{at\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t}$$

Radius of Curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left[1+\tan^2 t\right]^{3/2}}{\frac{1}{at\cos^3 t}}$$

$$= \left[1+\tan^2 t\right]^{3/2} \times at\cos^3 t$$

$$= \left[1+\frac{\sin^2 t}{\cos^2 t}\right]^{3/2} \times at\cos^3 t$$

$$= \left[\frac{\sin^2 t+\cos^2 t}{\cos^2 t}\right]^{3/2} \times at\cos^3 t$$

$$= \left[\frac{1}{\cos^2 t}\right]^{3/2} \times at\cos^3 t$$

$$= \frac{1}{\cos^3 t} \times at\cos^3 t$$

$$\rho = at$$

$$\boxed{\therefore \rho = at}$$

Problem - 6

Find the radius of curvature at the point θ on the curve $x = a\log\sec\theta$ and $y = a(\tan\theta - \theta)$

Soln:

Gin Curve ; $x = a\log\sec\theta$

$y = a(\tan\theta - \theta)$

Differentiating w.r.to θ, we get

$$x' = a\frac{1}{\sec\theta}\sec\theta\tan\theta = a\tan\theta$$

$$y' = a(\sec^2\theta - 1) = a\tan^2\theta$$

$$x'' = a\sec^2\theta$$

$$y'' = 2a\tan\theta(\sec^2\theta)$$

Radius of Curvature $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$

$$= \frac{(a^2\tan^2\theta + a^2\tan^4\theta)^{3/2}}{(a\tan\theta)(2a\tan\theta\sec^2\theta) - (a\tan^2\theta)(a\sec^2\theta)}$$

$$= \frac{\left[a^2\tan^2\theta(1+\tan^2\theta)\right]^{3/2}}{2a^2\tan^2\theta\sec^2\theta - a^2\tan^2\theta\sec^2\theta}$$

$$= \frac{\left[a^2\tan^2\theta\cdot\sec^2\theta\right]^{3/2}}{a^2\tan^2\theta\sec^2\theta}$$

$$= \frac{a^3\tan^3\theta\sec^3\theta}{a^2\tan^2\theta\sec^2\theta}$$

$$\boxed{\rho = a\tan\theta\sec\theta}$$

Pblm - 7

P.T the radius of curvature of the Catenary uniform strength $y = a\log\sec\left(\frac{x}{a}\right)$ is $a\sec\frac{x}{a}$.

Soln:

Gn Curve $y = a\log\sec\left(\frac{x}{a}\right)$

Differentiating w.r.to x, we get

$$\frac{dy}{dx} = \cancel{a}\frac{1}{\sec(x/a)}\cdot\sec(x/a)\cdot\tan(x/a)\cdot\left(\frac{1}{\cancel{a}}\right)$$

$$\frac{dy}{dx} = \tan(x/a)$$

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{x}{a}\right)\cdot\frac{1}{a} = \frac{1}{a}\sec^2(x/a)$$

Radius of Curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\tan^2(x/a)\right]^{3/2}}{1/a \sec^2 x/a}$$

$$= \frac{a\left[\sec^2(x/a)\right]^{3/2}}{\sec^2(x/a)}$$

$$= \frac{a \sec^{\cancel{3}}(x/a)}{\cancel{\sec^2 x/a}}$$

$$\boxed{\rho = a \sec(x/a)}$$

Pbm - 8

Find ρ for the curve $4ay^2 = (2a-x)^3$ at $(a, a/2)$.

Soln:

Differentiating w.r to x we get

$$8ay \frac{dy}{dx} = 3(2a-x)^2(-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3(2a-x)^2}{8ay} = \frac{-3}{8a} \frac{(2a-x)^2}{y}$$

$$\left(\frac{dy}{dx}\right)_{(a,\, a/2)} = \frac{-3(2a-a)^2}{\cancel{4}8a(a/2)}$$

$$= \frac{-3a^2}{4a^2} = \frac{-3}{4}$$

$$\frac{d^2y}{dx^2} = \frac{-8ay\left(6(2a-x)(-1)\right) + 3(2a-x)^2 \, 8a\frac{dy}{dx}}{(8ay)^2}$$

$$= \frac{8ay\left[6(2a-x)(1)\right] + 24a(2a-x)^2\frac{dy}{dx}}{64a^2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{8ay\left[6(2a-x)\right] + 24a(2a-x)^2\left(-\frac{3}{4}\right)}{64a^2y^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(a,\,a/2)} = \frac{8a\left(\frac{a}{2}\right)\left[6(2a-a)\right] + 24a(2a-a)^2\left(-\frac{3}{4}\right)}{64a^2\left(\frac{a^2}{4}\right)}$$

$$= \frac{4a^2(6a) - 18a(a^2)}{16a^4}$$

$$= \frac{24a^3 - 18a^3}{16a^4}$$

$$= \frac{6a^3}{16a^4}$$

$$= \frac{3}{8a}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + \frac{9}{16}\right]^{3/2}}{3/8a}$$

$$= \frac{8a}{3}\left(\frac{25}{16}\right)^{3/2}$$

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$$= \frac{8a}{3}\left[\left(\frac{5}{4}\right)^2\right]^{3/2}$$

$$= \frac{8a}{3} \times \frac{125}{64}$$

$$\therefore \boxed{\rho = \frac{125a}{24}}$$

H.W

1) $\sqrt{x} + \sqrt{y} = 1$ at the point $(\frac{1}{4}, \frac{1}{4})$ $\rho = ?$

Solution:

Gn Curve : $\sqrt{x} + \sqrt{y} = 1$

Differentiating with respect to x, we get,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-2\sqrt{y}}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\left(\frac{dy}{dx}\right)_{(1/4,\,1/4)} = \frac{-\sqrt{1/4}}{\sqrt{1/4}} = \frac{-1/2}{1/2}$$

$$\frac{dy}{dx} = -1$$

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} + \sqrt{y}\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{\frac{\sqrt{x}}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1/4,\,1/4)} = \frac{\frac{1/2}{\cancel{2}\times 1/\cancel{2}} + \frac{1/2}{\cancel{2}\times 1/\cancel{2}}}{1/4}$$

$$= \frac{\frac{1}{2}+\frac{1}{2}}{1/4}$$

$$= \frac{\cancel{2}}{\cancel{2}} \times 4$$

$$\frac{d^2y}{dx^2} = 4$$

Radius of Curvature $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

$$= \frac{\left[1+(-1)^2\right]^{3/2}}{4}$$

$$= \frac{(2)^{3/2}}{4} = \frac{(8)^{1/2}}{4}$$

$$= \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{\sqrt{\cancel{2}}}{\cancel{\sqrt{2}}\sqrt{2}}$$

$$\therefore \boxed{\rho = \frac{1}{\sqrt{2}}}$$

H.w 2) $y^2 = x^3 + 8$ at the point $(-2, 0)$ $\rho = ?$

Soln:

Gn Curve : $y^2 = x^3 + 8$

Differentiating w. r. to x. we get,

$$2y = 3x^2 \frac{dx}{dy} + 0$$

$$\frac{dx}{dy} = \frac{2y}{3x^2}$$

$$\left.\frac{dx}{dy}\right|_{(-2,0)} = \frac{2(0)}{3(-2)^2} = 0$$

$$\frac{d^2x}{dy^2} = \frac{3x^2(2) - 2y\left(6x \frac{dx}{dy}\right)}{9x^4}$$

$$= \frac{6x^2 - 12xy(0)}{9x^4} = \frac{6x^2}{9x^4}$$

$$= \frac{2}{3x^2}$$

$$\left.\frac{d^2x}{dy^2}\right|_{(-2,0)} = \frac{2}{3(4)} = \frac{1}{6}$$

Radius of Curvature $\rho = \dfrac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$

$$= \frac{(1+0)^{3/2}}{1/6} = \frac{(1)^{3/2}}{1/6}$$

$$\therefore \boxed{\rho = 6}$$

H.3/3) $xy = 30$ at $(3, 10)$ $\rho = ?$

Soln:

Gn Curve : $xy = 30$

Differentiating with respect to x. we get

$$x\frac{dy}{dx} + y(1) = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx}_{(3,10)} = \frac{-10}{3}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y(1)}{x^2} = \frac{x\left(\frac{10}{3}\right) + y}{x^2}$$

$$\frac{d^2y}{dx^2}_{(3,10)} = \frac{\cancel{3}\left(\frac{10}{\cancel{3}}\right) + 10}{9} = \frac{20}{9}$$

Radius of Curvature $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

$$= \frac{\left[1 + \left(-10/3\right)^2\right]^{3/2}}{20/9}$$

$$= \frac{\left[1 + \frac{100}{9}\right]^{3/2}}{20/9}$$

$$= \frac{\left[109/9\right]^{3/2}}{20/9}$$

$$= \frac{(109)^{3/2}}{(3)^3} \times \frac{9}{20}$$

$$= \frac{(109)^{3/2}}{\cancel{9} \times 3} \times \frac{\cancel{9}}{20}$$

$$\therefore \rho = \frac{(109)^{3/2}}{60}$$

4.(4) $xy^3 = a^4$ at the point (a, a), $\rho = ?$

Soln:

Gn Curve : $xy^3 = a^4$

Differentiating w.r.to x we get,

$$x\left(3y^2 \frac{dy}{dx}\right) + y^3(1) = 0$$

$$\frac{dy}{dx} = \frac{-y^{\cancel{3}}}{3x\cancel{y^2}}$$

$$\frac{dy}{dx} = \frac{-y}{3x}$$

$$\frac{dy}{dx}_{(a,a)} = \frac{-\cancel{a}}{3\cancel{a}} = \frac{-1}{3}$$

$$\frac{d^2y}{dx^2} = \frac{-3x\frac{dy}{dx} + y(3)}{9x^2} = \frac{\cancel{x}\cancel{3}(\frac{1}{\cancel{3}}) + 3y}{9x^2}$$

$$= \frac{x + 3y}{9x^2}$$

$$\frac{d^2y}{dx^2}_{(a,a)} = \frac{a + 3a}{9a^2} = \frac{4\cancel{a}}{9a^{\cancel{2}}} = \frac{4}{9a}$$

Radius of Curvature $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$= \frac{\left[1 + \frac{1}{9}\right]^{3/9}}{4/9a}$$

$$= \left[\frac{10}{9}\right]^{3/2} \frac{9a}{4}$$

$$= \frac{(10)^{3/2}}{(3)^3} \cdot \frac{9a}{4}$$

$$= \frac{(10)^{3/2}}{9 \times 3} \cdot \frac{9a}{4}$$

$$\rho = \frac{(10)^{3/2}}{12} a$$

Pbm - 9

P.T the radius of Curvature at a point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.

Soln

Gin Curve $x^{2/3} + y^{2/3} = a^{2/3}$

Diff w.r to x. we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}}$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$$

$$\left(\frac{dy}{dx}\right)_{(a\cos^3\theta,\, a\sin^3\theta)} = \frac{-(a\sin^3\theta)^{1/3}}{(a\cos^3\theta)^{1/3}}$$

$$= \frac{-a^{1/3}\sin\theta}{a^{1/3}\cos\theta} = -\tan\theta$$

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$$\frac{d^2y}{dx^2} = \frac{-x^{1/3}\left(\frac{1}{3}y^{-2/3}\frac{dy}{dx}\right) + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{\left(x^{1/3}\right)^2}$$

$$= \frac{\frac{1}{3}\left[x^{-2/3}y^{1/3} - x^{1/3}y^{-2/3}\frac{dy}{dx}\right]}{x^{2/3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(a\cos^3\theta,\, a\sin^3\theta)} = \frac{\frac{1}{3}\left[(a^{-2/3}\cos^{-2}\theta)(a^{1/3}\sin\theta) - (a^{1/3}\cos\theta)(a^{-2/3}\sin^{-2}\theta)(-\tan\theta)\right]}{a^{2/3}\cos^2\theta}$$

$$= \frac{\frac{1}{3}\left[a^{-1/3}\frac{\sin\theta}{\cos^2\theta} + a^{-1/3}\frac{\cos\theta}{\sin^2\theta}\cdot\frac{\sin\theta}{\cos\theta}\right]}{a^{2/3}\cos^2\theta}$$

$$= \frac{\frac{1}{3}\left[\frac{\sin\theta}{\cos^2\theta} + \frac{1}{\sin\theta}\right]}{a^{2/3+1/3}\cos^2\theta}$$

$$= \frac{\frac{1}{3}\left[\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin\theta}\right]}{a\cos^2\theta}$$

$$= \frac{\frac{1}{3}\times\frac{1}{\cos^2\theta\sin\theta}}{a\cos^2\theta}$$

$$= \frac{1}{3a\cos^4\theta\sin\theta}$$

Radius of Curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\tan^2\theta\right)^{3/2}}{\frac{1}{3a\cos^4\theta\sin\theta}}$$

$$= \left[1 + \frac{\sin^2\theta}{\cos^2\theta}\right]^{3/2} 3a\cos^4\theta\sin\theta$$

$$= \left[\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right]^{3/2} 3a\cos^4\theta\sin\theta$$

$$= \frac{1}{\cos^3\theta}\, 3a\cos^4\theta\sin\theta$$

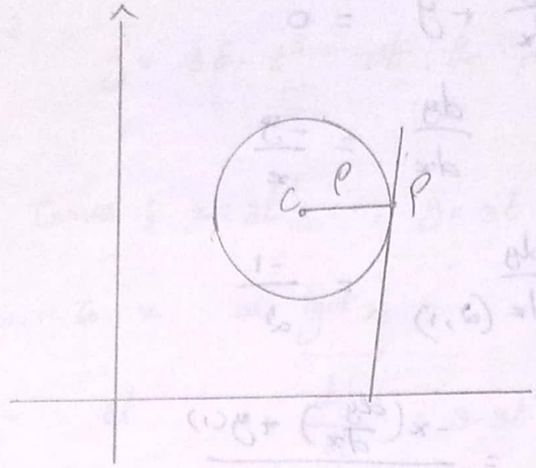
$$\rho = 3a\cos\theta\sin\theta$$

~~Centers~~ Center and Circle of Curvature

Deffinition

Consider a point P on any gn curve. Draw the normal to the curve at P. Let C be the point on the normal to the curve at P such that CP = ρ and C lies on the side towards which the curve is concave.

The C is called center of curvature to the curve at P. The circle with center C and radius ρ is called the circle of curvature.



Co-ordinates of Center of Curvature.

Let $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

The co-ordinates of the center of curvature is

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \;\&$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

Pbm - 10

Find the co-ordinates of the center of curvature of the curve $xy = 2$ at the pt $(2, 1)$

Soln:

Gn curve : $xy = 2$

Diff w.r.to x, we get.

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx}_{(2,1)} = \frac{-1}{2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\left(\frac{dy}{dx}\right) + y(1)}{x^2}$$

$$= \frac{-x\frac{dy}{dx} + y}{x^2} = \frac{-x\left(\frac{-y}{x}\right) + y}{x^2}$$

$$= \frac{2y}{x^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(2,1)} = \frac{2}{4} = \frac{1}{2}$$

$$y_1 = \frac{-1}{2}, \quad y_2 = \frac{1}{2}$$

The Co-ordinates of Center of Curvature.

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 2 - \frac{(-\frac{1}{2})(1+\frac{1}{4})}{\frac{1}{2}}$$

$$= 2 + \cancel{\tfrac{1}{2}}\left(\frac{5}{4}\right) \times \left(\frac{\cancel{2}}{1}\right)$$

$$= 2 + \frac{5}{4} = \frac{13}{4} = 3\frac{1}{4}$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$= 1 + \frac{1+\frac{1}{4}}{\frac{1}{2}}$$

$$= 1 + \left(\frac{5}{\cancel{4}_2} \times \cancel{2}\right)$$

$$= 1 + \frac{5}{2}$$

$$= \frac{7}{2} = 3\frac{1}{2}$$

The co-ordinates of Center of Curvature is $\left(3\frac{1}{4}, 3\frac{1}{2}\right)$

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5) $x = 3t^2$. $y = 3t - t^3$ at the pt $t=1$ $\rho = ?$

Soln :

Gn Curve ; $x = 3t^2$, $y = 3t - t^3$

Diff w.r to x, we get,

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 3 - 3t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{3-3t^2}{6t} = \frac{\cancel{3}(1-t^2)}{\cancel{6}_2 t} = \frac{1-t^2}{2t}$$

$$\frac{dy}{dx}_{(t=1)} = \frac{1-1}{2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1-t^2}{2t}\right)\frac{1}{6t}$$

$$= \frac{2t(-2t) - (1-t^2)(2)}{4t^2} \times \frac{1}{6t}$$

$$= \frac{-4t^2 - 2 + 2t^2}{24t^3}$$

$$= \frac{-2t^2 - 2}{24t^3}$$

$$= \frac{-\cancel{2}(t^2+1)}{\cancel{24}t^3} = \frac{-(t^2+1)}{12t^3}$$

$$\frac{d^2y}{dx^2}_{(t=1)} = \frac{-(1+1)}{12(1)} = \frac{-2}{12} = \frac{-1}{6}$$

Radius of Curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{(1+0)^{3/2}}{-1/6} = 1 \times -6$$

$$\therefore \rho = -6$$

H.W

6) $x = a(\cos t + \sin t)$ $y = a(\cos t - \sin t)$ at the point $\rho = ?$

Soln:

Gn Curve. $x = a(\cos t + \sin t)$

$y = a(\cos t - \sin t)$

Diff w.r.to x. we get,

$x' = a(-\sin t + \cos t) = a(\cos t - \sin t)$

$y' = a(-\sin t - \cos t) = -a(\cos t + \sin t)$

$x'' = a(-\sin t - \cos t) = -a(\cos t + \sin t)$

$y'' = -a(-\sin t + \cos t) = -a(\cos t - \sin t)$

Radius of Curvature

$$\rho = \frac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{[a^2(\cos t - \sin t)^2 + a^2(\cos t + \sin t)^2]^{3/2}}{-a^2(\cos t - \sin t)^2 - a^2(\cos t + \sin t)^2}$$

$a^{3/2}$

$\cancel{z}$

$$= \frac{a^3\left[\cos^2 t+\sin^2 t-2\cancel{\cos t\sin t}+\cos^2 t+\sin^2 t+2\cancel{\cos t\sin t}\right]^{3/2}}{-a^2\left[(\cos t-\sin t)^2+(\cos t+\sin t)^2\right]}$$

$$= \frac{a^3(1+1)^{3/2}}{-a^2\left[\sin^2 t-2\cos t\sin t+\cos^2 t+\cos^2 t+\sin^2 t+4\sin t\cos t\right]}$$

$$= \frac{a(2)^{3/2}}{-4\cos t\sin t} = \frac{a(8)^{1/2}}{-4\cos t\sin t}$$

$$= \frac{a\,\cancel{2}\sqrt{2}}{-\cancel{4}\cos t\sin t} = \frac{a\sqrt{\cancel{2}}}{-\sqrt{\cancel{2}}\sqrt{2}\sin t\cos t}$$

$$\therefore \boxed{\rho = \frac{-a}{\sqrt{2}\sin t\cos t}}$$

Ans $= \frac{a}{\sqrt{2}}$

7) H.W. S.T $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a,0)$

$\rho = \frac{a}{2}$

Soln:

Diff w.r.t y. we get

$$2y = \frac{x\left[a^2\left(-\frac{dx}{dy}\right)\right]-a^2(a-x)\frac{dx}{dy}}{x^2}$$

$$2y = \frac{-\cancel{a^2x}\frac{dx}{dy}-a^3\frac{dx}{dy}+\cancel{a^2x}\frac{dx}{dy}}{x^2}$$

$$2x^2y = -a^3\frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{-2x^2y}{a^3}$$

$$\frac{dx}{dy}_{(a,0)} = 0$$

$$\frac{d^2x}{dy^2} = \frac{-2}{a^3}\left[x^2(1) + y\, 2x \frac{dx}{dy}\right]$$

$$= \frac{-2}{a^3}\left[x^2 + 2xy \frac{dx}{dy}\right]$$

$$\frac{d^2x}{dy^2}_{(a,0)} = \frac{-2}{a^3}\left[a^2 + 2(a)(0)\,0\right]$$

$$= \frac{-2}{a^{\cancel{3}}} (\cancel{a^2})$$

$$= \frac{-2}{a}$$

Radius of Curvature

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2}$$

$$= \frac{[1+0]^{3/2}}{-2/a}$$

$$= \frac{-a}{2}$$

$$\boxed{\rho = \frac{-1}{2} a}$$

H.8/ 8) $x^3 + y^3 + 2x^2 - 4y + 3x = 0$ at the point origin

$\rho = ?$

Soln:

Gn Curve : $x^3 + y^3 + 2x^2 - 4y + 3x = 0$

Diff w.r.to x. we get,

$$3x^2 + 3y^2 \frac{dy}{dx} + 4x - 4\frac{dy}{dx} + 3 = 0$$

$$3y^2 \frac{dy}{dx} - 4\frac{dy}{dx} = -3x^2 - 4x - 3$$

$$\frac{dy}{dx}(3y^2 - 4) = -3x^2 - 4x - 3$$

$$\frac{dy}{dx} = \frac{-3x^2 - 4x - 3}{3y^2 - 4}$$

$$\frac{dy}{dx}_{(0,0)} = \frac{0+0-3}{0-4}$$

$$= \frac{-3}{-4}$$

$$\frac{dy}{dx} = 3/4$$

$$\frac{d^2y}{dx^2} = \frac{(3y^2-4)(-6x-4) + (3x^2+4x+3)(6y\frac{dy}{dx})}{(3y^2-4)^2}$$

$$\frac{d^2y}{dx^2}_{(0,0)} = \frac{(0-4)(0-4) + (0+0+3)(0)}{(0-4)^2}$$

$$= \frac{(-4)(-4)}{(-4)^2} = \frac{16}{16} = 1$$

Radius of Curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left[1 + (3/4)^2\right]^{3/2}}{1} = \left[1 + \frac{9}{16}\right]^{3/2} = \left[\frac{25}{16}\right]^{3/2}$$

$$= \left[\frac{5}{4}\right]^3 = \frac{125}{64}$$

$$\therefore \boxed{\rho = \frac{125}{64}}$$

Pbm - 4

S.T in the Parabola $y^2 = 4ax$ at the point t, $\rho = -2a(1+t^2)^{3/2}$

$x = 2a + 3at^2$, $y = -2at^3$

Soln:

W.K.T the parametric eqn of the parabola $x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2a \frac{1}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{-1}{t^2} \times \frac{1}{2at}$$

$$= \frac{-1}{2at^3}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left[1 + \frac{1}{t^2}\right]^{3/2}}{\frac{-1}{2at^3}}$$

$$= -\left[1 + \frac{1}{t^2}\right]^{3/2} \times 2at^3$$

$$= \frac{-(t^2+1)^{3/2}}{(t^2)^{3/2}} \, 2at^3$$

$$= \frac{-(t^2+1)^{3/2}}{t^3} \, 2at^3$$

$$= -(1+t^2)^{3/2} \times 2a$$

$$\therefore \boxed{\rho = -2a(1+t^2)^{3/2}}$$

The Co-ordinats of Center of Curvature.

$$x = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$$= at^2 + \frac{2at^3}{t}\left(\frac{t^2+1}{t^2}\right)$$

$$= at^2 + 2a(t^2+1)$$

$$= at^2 + 2at^2 + 2a$$

$$= 3at^2 + 2a$$

$$\boxed{x = 2a + 3at^2}$$

$$y = y + \frac{1+y_1^2}{y_2} = 2at + \frac{1+\frac{1}{t^2}}{\frac{-1}{2at}}$$

$$= 2at - 2at^3\left(\frac{t^2+1}{t^2}\right)$$

$$= 2at - 2at(t^2+1)$$

$$= 2at - 2at^3 - 2at$$

$$\boxed{y = -2at^3}$$

Pbm - 12

S.T for a curve $x^{2/3} + y^{2/3} = a^{2/3}$

$x = a\cos^3 t + 3a\cos t\sin^2 t$

$y = a\sin^3 t + 3a\sin t\cos^2 t$

Soln:

W.K.T the parametric eqn of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is

$x = a\cos^3 t \quad , y = a\sin^3 t$

$$\frac{dx}{dt} = a\,3\cos^2 t(-\sin t) = -3a\sin t\cos^2 t$$

$$\frac{dy}{dt} = a\,3\sin^2 t\cos t = 3a\sin^2 t\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{3a\sin^2 t\cos t}{-3a\sin t\cos^2 t}$$

$$= \frac{-\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\left(\frac{dt}{dx}\right)$$

$$= \frac{d}{dt}(-\tan t)\frac{1}{-3a\sin t\cos^2 t}$$

$$= -\sec^2 t\,\frac{1}{-3a\sin t\cos t}$$

$$= \frac{\frac{1}{\cos^2 t}}{3a\sin t\cos t}$$

$$= \frac{1}{3a\sin t\cos^4 t}$$

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The Co-ordinates of Center of Curvature

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos^3 t + \frac{\tan t\,(1+\tan^2 t)}{\frac{1}{3a\sin t\cos^4 t}}$$

$$= a\cos^3 t + \tan t \sec^2 t \times 3a\sin t\cos^4 t$$

$$= a\cos^3 t + \tan t \frac{1}{\cos^2 t} \times 3a\sin t\cos^4 t$$

$$= a\cos^3 t + 3a\sin t\cos^2 t\,\tan t$$

$$= a\cos^3 t + 3a\sin t\cos^2 t\,\frac{\sin t}{\cos t}$$

$$\boxed{X = a\cos^3 t + 3a\sin^2 t\cos t}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= a\sin^3 t + \frac{(1+\tan^2 t)}{\frac{1}{3a\cos^4 t\sin t}}$$

$$= a\sin^3 t + \sec^2 t \times 3a\cos^4 t\sin t$$

$$= a\sin^3 t + \frac{1}{\cos^2 t} \times 3a\sin t\cos^4 t$$

$$\boxed{Y = a\sin^3 t + 3a\sin t\cos^2 t}$$

Pbm - 18

S.T is ~~it~~ a parabola

$\sqrt{x} + \sqrt{y} = \sqrt{a}$ ~~x+y~~ $X+Y = 3(x+y)$

Soln:

Gn Curve :

$\sqrt{x} + \sqrt{y} = \sqrt{a}$

Diff. w.r.to x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2\sqrt{x}} \times 2\sqrt{y}$$

$$= \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} + \sqrt{y}\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{\frac{\sqrt{x}}{2\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{x}} + \frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{1}{2} + \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}}}{x}$$

$$= \frac{1}{2}\left(1 + \frac{\sqrt{y}}{\sqrt{x}}\right)\frac{1}{x}$$

$$= \frac{1}{2x}\left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\right)$$

$$= \frac{1}{2x}\left(\frac{\sqrt{a}}{x^{1/2}}\right)$$

$$= \frac{1}{2x^{3/2}}\sqrt{a}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{a}}{2x^{3/2}}$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x + \frac{\sqrt{y}/\sqrt{x}\left(1+\frac{y}{x}\right)}{\frac{\sqrt{a}}{2x^{3/2}}}$$

$$= x + \frac{\sqrt{y}}{\sqrt{x}}\left(\frac{x+y}{x}\right)\frac{2x^{3/2}}{\sqrt{a}}$$

$$\boxed{X = x + \frac{2\sqrt{y}}{\sqrt{a}}(x+y)}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= y + \frac{1+\frac{y}{x}}{\sqrt{a}/2x^{3/2}}$$

$$= y + \left(1+\frac{y}{x}\right)\times\frac{2x^{3/2}}{\sqrt{a}}$$

$$= y + \left(\frac{x+y}{x}\right)\times\frac{2x^{3/2}}{\sqrt{a}}$$

$$\boxed{Y = y + \frac{2\sqrt{x}}{\sqrt{a}}(x+y)}$$

Now,

$$X+Y = x + \frac{2\sqrt{y}}{\sqrt{a}}(x+y) + y + \frac{2\sqrt{x}}{\sqrt{a}}(x+y)$$

$$= (x+y) + \frac{2}{\sqrt{a}}(\sqrt{x}+\sqrt{y})(x+y)$$

$$= (x+y) + \frac{2}{\sqrt{a}}(x+y)(\sqrt{a})$$

$$= x+y+2(x+y)$$

$$\boxed{X+Y = 3(x+y)}$$

Pbm - 14

S.T the eqn of the circle of curvature at the origin of the parabola $y = mx + \frac{x^2}{a}$ is

$$x^2+y^2 = (1+m^2)\, a\, (y - mx)$$

Soln:

Gn curve :

$$y = mx + \frac{x^2}{a}$$

Diff w.r. to x, we get

$$\frac{dy}{dx} = m + \frac{2x}{a}$$

$$\left(\frac{dy}{dx}\right)_{(0,0)} = m$$

$$\frac{d^2y}{dx^2} = \frac{2}{a}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,0)} = 2/a$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 0 - \frac{m(1+m^2)}{2/a}$$

$$\boxed{X = -\frac{am}{2}(1+m^2)}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= 0 + \frac{1+m^2}{2/a} = \frac{a}{2}(1+m^2)$$

$$\boxed{Y = a/2\,(1+m^2)}$$

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$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left[1+m^2\right]^{3/2}}{2/a}$$

$$\boxed{\rho = \frac{a}{2}\cdot(1+m^2)^{3/2}}$$

The eqn of circle of curvature is

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$\left(x + \frac{am(1+m^2)}{2}\right)^2 + \left(y - \frac{a(1+m^2)}{2}\right)^2 = \left(\frac{a(1+m^2)^{3/2}}{2}\right)^2$$

$$x^2 + \frac{a^2m^2(1+m^2)^2}{4} + 2(x)\left(\frac{am(1+m^2)}{2}\right) + y^2 + \frac{a^2(1+m^2)^2}{4} - 2(y)\left(\frac{a(1+m^2)}{2}\right) = \frac{a^2(1+m^2)^3}{4}$$

$$x^2 + \frac{a^2m^2}{4}(1+m^2)^2 + amx(1+m^2) + y^2 + \frac{a^2}{4}(1+m^2)^2 - ay(1+m^2) = \frac{a^2}{4}(1+m^2)^3$$

$$\Rightarrow x^2+y^2 = \frac{a^2}{4}(1+m^2)^3 - \frac{a^2m^2}{4}(1+m^2)^2 - max(1+m^2) - \frac{a^2}{4}(1+m^2)^2 + ay(1+m^2)$$

$$\Rightarrow x^2+y^2 = (1+m^2)\left(\frac{a^2}{4}(1+m^2)^2 - \frac{a^2}{4}(1+m^2)(1+m^2) + ay - max\right)$$

$$\Rightarrow x^2+y^2 = (1+m^2)\left(\frac{a^2}{4}(1+m^2)^2 - \frac{a^2}{4}(1+m^2)^2 + a(y-mx)\right)$$

$$\Rightarrow x^2+y^2 = (1+m^2)\left(a(y \mp mx)\right)$$

$$x^2+y^2 = (1+m^2)a(y-mx)$$

H.W

Co-ordinates of Center of curvature

1) $y = x^2 \qquad (1/2, 1/4)$

Soln

Gin Curve : $y = x^2$

Diff w.r to x, we get,

$$\frac{dy}{dx} = 2x$$

$$\left(\frac{dy}{dx}\right)_{(1/2, 1/4)} = 2(1/2)$$

$$\frac{d^2y}{dx^2} = 2$$

$$\frac{d^2y}{dx^2}_{(1/2, 1/4)} = 2$$

Co-ordinates of Center of curvature :

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= \frac{1}{2} - \frac{1(1+1)}{2}$$

$$= \frac{1}{2} - \frac{2}{2}$$

$$= \frac{1}{2} - 1.$$

$$\boxed{X = \frac{-1}{2}}$$

$$Y = y + \frac{1+y_1^2}{y_2} = \frac{1}{4} + \frac{1+1}{2}$$

$$= \frac{1}{4} + \frac{2}{2} = \frac{1}{4} + 1$$

$$\boxed{Y = \frac{5}{4}}$$

Co-ordinates of center of curvature is $\left(\frac{-1}{2}, \frac{5}{4}\right)$

H.W 2) $xy = c^2$ (c, c)

Soln:

Gn curve $xy = c^2$

Diff w.r to x. we get

$$x \cdot \frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx}_{(c,c)} = \frac{-c}{c} = -1$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y}{x^2}$$

$$= \frac{-x(-1) + y}{x^2} = \frac{x+y}{x^2}$$

$$\frac{d^2y}{dx^2}_{(c,c)} = \frac{2c}{c^2} = \frac{2}{c}$$

Co-ordinates of Center of Curvature :

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= c + \frac{1(1+1)}{2/c}$$

$$= c + \frac{2}{2/c}$$

$$= c + \frac{\cancel{2}c}{\cancel{2}}$$

$$\boxed{X = 2c}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$= c + \frac{(1+1)}{2/c}$$

$$= c + \cancel{2} \times \frac{c}{\cancel{2}}$$

$$= c + c$$

$$\boxed{Y = 2c}$$

Co-ordinates of center of curvature.

$(2c, 2c)$

3) $x = 3\cos t + \cos 3t$ $t=0$

H.W

$y = 3\sin t - \sin 3t$

Soln.

Gn Curve $x = 3\cos t + \cos 3t$

$y = 3\sin t - \sin 3t$

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$$\frac{dx}{dt} = -3\sin t - \sin 3t\,(3)$$

$$= -3\sin t - 3\sin 3t = -3(\sin t + \sin 3t)$$

$$\frac{dy}{dt} = 3\cos t - \cos 3t\,(3) = 3(\cos t - \cos 3t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{\cancel{3}(\cos t - \cos 3t)}{-\cancel{3}(\sin t + \sin 3t)}$$

$$= \frac{-(\cos t - \cos 3t)}{\sin t + \sin 3t}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = 0 \qquad \frac{d^2y}{dx^2} = 0$$

$$\cancel{x_{t=0}}\ x_{t=0} = 3 + 1 = 4$$

$$y_{t=0} = 0$$

Co-ordinates of Center of Curvature:

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 4 - 0$$

$$\boxed{X = 4}$$

$$Y = y + \frac{y_1(1+y_1^2)}{y_2}$$

$$= 0 + 0$$

$$\boxed{Y = 0}$$

Co-ordinates of Center of Curvature:

(4, 0)

Radius of Curvature when the curve gn in the polar co-ordinates

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

Pbm - 15

Find the radius of curvature of the cardiod $r = a(1-\cos\theta)$

Soln

Gin Curve : $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a(+\sin\theta) = a\sin\theta$$

$$\frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2} = \left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta\right]^{3/2}$$

$$= \left\{a^2\left[1 - 2\cos\theta + \sin^2\theta + \cos^2\theta\right]\right\}^{3/2}$$

$$= \left[a^2(2 - 2\cos\theta)\right]^{3/2}$$

$$= a^3\left[2(1-\cos\theta)\right]^{3/2}$$

$$\cancel{= 8a^3\left[2\sin^2\frac{\theta}{2}\right]^{3/2}}$$

$$\cancel{= 8a^3}$$

$$= a^3\left[2(2\sin^2\theta/2)\right]^{3/2}$$

$$= a^3\left[\left(2\sin^2\theta/2\right)^2\right]^{3/2}$$

$$= 8a^3 \sin^3 \theta/2$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}$$

$$= a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1-\cos\theta)\cos\theta$$

$$= a^2(1+\cos^2\theta - 2\cos\theta) + 2a^2\sin^2\theta - a^2(\cos\theta - \cos^2\theta)$$

$$= a^2\left[1+\cos^2\theta - 2\cos\theta + \sin^2\theta + \sin^2\theta - \cos\theta + \cos^2\theta\right]$$

$$= a^2(3 - 3\cos\theta)$$

$$= 3a^2(1-\cos\theta)$$

$$= 3a^2(2\sin^2\theta/2)$$

$$= 6a^2\sin^2\theta/2$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{8a^3\sin^3\theta/2}{6a^2\sin^2\theta/2}$$

$$= \frac{4}{3}a\sin\theta/2$$

$$= \frac{4}{3}a\left(\frac{1-\cos\theta}{2}\right)^{1/2}$$

$$= \frac{2\times\sqrt{2}\times\sqrt{2}}{3\times\sqrt{2}}a(1-\cos\theta)^{\frac{1}{2}}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{a} \cdot \sqrt{a}(1-\cos\theta)^{1/2}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{a}\sqrt{r}$$

$$\boxed{\rho = \frac{2}{3}\sqrt{2ar}}$$

Pbm - 16

Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ is $\frac{a^n r^{1-n}}{n+1}$

Soln:

Given curve : $r^n = a^n \cos n\theta$

Taking log on both sides, we get,

$$n \log r = \log(a^n \cos n\theta)$$

$$\Rightarrow n \log r = \log a^n + \log \cos n\theta$$

$$\Rightarrow n \log r = n \log a + \log \cos n\theta$$

Diff w.r to θ, we get

$$\Rightarrow \frac{n}{r}\frac{dr}{d\theta} = \frac{1}{\cos n\theta}(n(-\sin n\theta))$$

$$\Rightarrow \frac{n}{r}\frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-r \sin n\theta}{\cos n\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r \tan n\theta$$

$$\frac{d^2r}{d\theta^2} = \frac{-d}{d\theta}(r \tan n\theta) \cancel{= -r\sec^2 n\theta}$$

$$-\frac{dr}{d\theta}\tan n\theta - rn\sec^2 n\theta$$

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{\left[r^2+r^2\tan^2 n\theta\right]^{3/2}}{r^2+2r^2\tan^2 n\theta - r\left[-\frac{dr}{d\theta}\tan\theta - nr\sec^2 n\theta\right]}$$

$$= \frac{\left[r^2(1+\tan^2 n\theta)\right]^{3/2}}{r^2+2r^2\tan^2 n\theta - r\left[r\tan^2 n\theta - nr\sec^2 n\theta\right]}$$

$$= \frac{r^3\sec^3 n\theta}{r^2+2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2+r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2(1+\tan^2 n\theta)+nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{\cancel{r^3}\sec^{\cancel{3}} n\theta}{(n+1)\cancel{r^2}\sec^{\cancel{2}} n\theta} \qquad r\frac{1}{\cos n\theta}$$

$$= \frac{r\sec n\theta}{(n+1)}$$

$$= \frac{r}{(n+1)\cos n\theta}$$

$$= \frac{r a^n}{(n+1)a^n\cos n\theta}$$

$$= \frac{r\, a^n}{r^n (n+1)}$$

$$= \frac{a^n r}{(n+1)} r^{-n}$$

$$= \frac{a^n}{(n+1)} r^{-n+1}$$

$$\rho = \frac{a^n}{n+1} r^{1-n}$$

Particular cases:

i) Putting $n=2$, we get Bernoulli's Lemniscate, $\rho = \frac{a^2}{3r}$

ii) When $n=-2$ we have rectangular hyperbola. $\rho = \frac{r^3}{a^2}$

iii) When $n = \frac{1}{2}$ we get cardiod $\rho = \frac{2}{3}\sqrt{ar}$

iv) When $n = \frac{-1}{2}$ we get parabola $\rho = \frac{2r^{3/2}}{\sqrt{a}}$

v) When $n=1$ we get circles $\rho = \frac{a}{2}$

Pbm - 17

The tangents at two Pts P, Q on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ are at right angles, ST If ρ and ρ' be the radius of curvature at the points $p^2 + p'^2 = 16a^2$

Soln

Gn curve $x = a(\theta - \sin\theta)$

$y = a(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\cancel{2a}\sin\theta/2 \cos\theta/2}{\cancel{2a}\sin^2\theta/2}$$

$$= \frac{\cos\theta/2}{\sin\theta/2} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\cot\theta/2 \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{2}\left(-\operatorname{cosec}^2\theta/2\right) \cdot \frac{1}{a(1-\cos\theta)}$$

$$= \frac{1}{2}\left(\frac{-1}{\sin^2\theta/2}\right) \cdot \frac{1}{a(1-\cos\theta)}$$

$$= \frac{1}{2}\left(\frac{-1}{\sin^2\theta/2}\right) \frac{1}{2a\sin^2\theta/2}$$

$$= \frac{-1}{4a\sin^4\theta/2}$$

$$= \frac{-1}{4a}\operatorname{cosec}^4\theta/2$$

At the pt P, θ_1 be the slope of the tangent is $\cot\left(\frac{\theta_1}{2}\right)$

At the pt Q, θ_2 be the slope of the tangent is $\cot\left(\frac{\theta_2}{2}\right)$

Since the tangents at P and Q are perpendicular. we have

$$\cot\left(\frac{\theta_1}{2}\right) \cot\left(\frac{\theta_2}{2}\right) = -1$$

$$\Rightarrow \frac{1}{\tan\left(\frac{\theta_1}{2}\right)} \frac{1}{\tan\left(\frac{\theta_2}{2}\right)} = -1$$

$$\Rightarrow \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\theta_2}{2}\right) = -1$$

$$\Rightarrow \frac{\sin(\theta_1/2)}{\cos(\theta_1/2)} \cdot \frac{\sin(\theta_2/2)}{\cos(\theta_2/2)} = -1$$

$$\Rightarrow \sin(\theta_1/2)\sin(\theta_2/2) + \cos(\theta_1/2)\cos(\theta_2/2) = 0$$

$$\Rightarrow \cos\left(\frac{\theta_1 - \theta_2}{2}\right) = 0$$

$$\Rightarrow \left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{\pi}{2}$$

$$\Rightarrow \theta_1 - \theta_2 = \pi$$

$$\Rightarrow \theta_2 = \pi - \theta_1 \quad (\text{Taking } \theta_2 > \theta_1)$$

Radius of curvature at P, θ_1 is

$$\rho = \frac{\left[1+\cot^2(\theta_1/2)\right]^{3/2}}{(-1/4a)\,\mathrm{cosec}^4(\theta_1/2)}$$

$$= \frac{\left[\mathrm{cosec}^2\,\theta_1/2\right]^{3/2}}{-\frac{1}{4a}\,\mathrm{cosec}^4\,\theta_1/2}$$

$$= \frac{-4a\,\mathrm{cosec}^3\,\theta_1/2}{\mathrm{cosec}^4\,\theta_1/2}$$

$$= \frac{-4a}{\mathrm{cosec}\,\theta_1/2} = -4a\sin(\theta_1/2)$$

Radius of curvature at Q, θ_2 is

$$\rho' = -4a\sin\theta_2/2$$

$$= -4a\sin(\pi/2 - \theta_1/2)$$

$$= -4a\cos\theta_1/2$$

$$\rho^2 + \rho'^2 = 16a^2\sin^2(\theta_1/2) + 16a^2\cos^2(\theta_1/2)$$

$$= 16a^2$$

$$\therefore \rho^2+\rho'^2 = 16a^2$$

Pbm-18

Prove that

$x = 3a\cos\theta - a\cos 3\theta$

$y = 3a\sin\theta - a\sin 3\theta$

$\rho = 3a\sin\theta.$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Soln:

Gin Curve : $x = 3a\cos\theta - a\cos 3\theta$

$y = 3a\sin\theta - a\sin 3\theta$

$x = 3a\cos\theta - a(4\cos^3\theta - 3\cos\theta)$

$= 3a\cos\theta - 4a\cos^3\theta + 3a\cos\theta$

$x = 6a\cos\theta - 4a\cos^3\theta$

$y = 3a\sin\theta - a(3\sin\theta - 4\sin^3\theta)$

$= \cancel{3a\sin\theta} - \cancel{3a\sin\theta} + 4a\sin^3\theta$

$y = 4a\sin^3\theta$

$x' = -6a\sin\theta + 12a\cos^2\theta\sin\theta$

$= 6a\sin\theta(2\cos^2\theta - 1)$

$\cos 2\theta = 2\cos^2\theta - 1$

$= 6a\sin\theta\cos 2\theta$

$x'' = 6a\cos\theta\cos 2\theta + 6a\sin\theta \cdot 2(-\sin 2\theta)$

$= 6a\cos\theta\cos 2\theta - 12a\sin\theta\sin 2\theta$

$y' = 12a\sin^2\theta \cdot \cos\theta$

$y'' = 12a(2\sin\theta\cos^2\theta) + 12a\sin^2\theta(-\sin\theta)$

$$= 24a\sin\theta\cos^2\theta - 12a\sin 3\theta$$

$$\left[(x')^2+(y')^2\right]^{3/2} = \left[36a^2\sin^2\theta\cos^2 2\theta + 144a^2\sin^4\theta\cos^2\theta\right]^{3/2}$$

$$= \left[36a^2\sin^2\theta\left(\cos^2 2\theta + 4\sin^2\theta\cdot\cos^2\theta\right)\right]^{3/2}$$

$$= \left[36a^2\sin^2\theta\left[\left(\cos^2\theta - \sin^2\theta\right)^2 + 4\sin^2\theta\cos^2\theta\right]\right]^{3/2}$$

$$\left[\cos 2\theta = \cos^2\theta - \sin^2\theta\right]$$

$$= \left[36a^2\sin^2\theta\left[\cos^4\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta + 4\sin^2\theta\cos^2\theta\right]\right]^{3/2}$$

$$= \left[36a^2\sin^2\theta\left(\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta\right)\right]^{3/2}$$

$$= \left[36a^2\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right)^2\right]^{3/2}$$

$$= \left[6a\sin\theta\left(\cos^2\theta + \sin^2\theta\right)\right]^3$$

$$= (6a\sin\theta)^3$$

$$x'y'' - y'x'' = 6a\sin\theta\cos 2\theta\left[24a\sin\theta\cos^2\theta - 12a\sin 3\theta\right]$$

$$- 12a\sin^2\theta\cos\theta\left[6a\cos\theta\cos 2\theta - 12a\sin\theta\sin 2\theta\right]$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= 6a\sin\theta\cos2\theta \cdot 12a\sin\theta\left[2\cos^2\theta - \sin^2\theta\right] -$$

$$12a\sin^2\theta\cos\theta \cdot 6a\cos\theta\left[\cos2\theta - 4\sin^2\theta\right]$$

$$= 72a^2\sin^2\theta\left[\cos2\theta\,(2\cos^2\theta - \sin^2\theta) - \cos^2\theta\,(\cos2\theta - 4\sin^2\theta)\right]$$

$$= 72a^2\sin^2\theta\left[2\cos^2\theta\cos2\theta - \sin^2\theta\cos2\theta - \cos^2\theta\cdot\cos2\theta + 4\sin^2\theta\cos^2\theta\right]$$

$$= 72a^2\sin^2\theta\left[\cos^2\cos2\theta - \sin^2\theta\cos2\theta + 4\sin^2\theta\cos^2\theta\right]$$

$$= 72a^2\sin^2\theta\left[\cos2\theta\left[\cos^2\theta - \sin^2\theta\right] + \sin^2 2\theta\right]$$

$$= 72a^2\sin^2\theta\left[\cos^2 2\theta + \sin^2 2\theta\right]$$

$$= 72a^2\sin^2\theta$$

$$\rho = \frac{(6a\sin\theta)^3}{72a^2\sin^2\theta}$$

$$= \frac{216\,a^3\sin^3\theta}{72\,a^2\sin^2\theta}$$

$$\rho = 3a\sin\theta$$

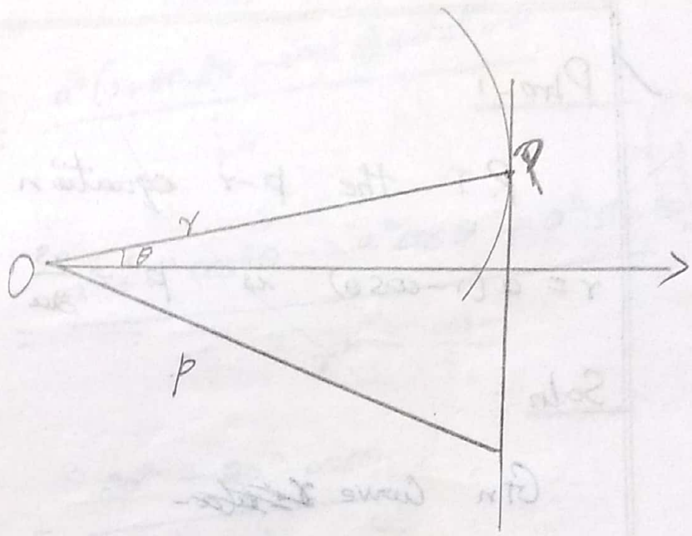
Unit- II

Pedal equation:

Let O be a Origin (or) pole.

Let P be any point on the curve.

Let p be the length of the perpendicular from O to the tangent at P.



Then $\boxed{\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2}$

The equation of a curve in terms of p and r is called pedal equation of the curve (or) simply p-r equation.

Remark:

1. If $r = \frac{1}{u}$, then $\dfrac{dr}{d\theta} = \dfrac{-1}{u^2}\dfrac{du}{d\theta}$

$\therefore$ The p-r equation becomes

$$\frac{1}{p^2} = u^2 + u^4\left(\frac{-1}{u^2}\frac{du}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = u^2 + u^4\left(\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2\right)$$

$$\Rightarrow \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

2. For any curve $\frac{ds}{d\theta} = \frac{r^2}{p}$

Pbm-1

P.T the p-r equation of the cardiod

$r = a(1-\cos\theta)$ is $p^2 = \frac{r^3}{2a}$

Soln:

Gn Curve ~~data~~ -

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$\therefore$ The p-r equation of a curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}a^2\sin^2\theta$$

$$\frac{1}{p^2} = \frac{r^2 + a^2\sin^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{2a^2 - 2a^2\cos\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{2a^2(1-\cos\theta)}{r^4}$$

$$\frac{1}{p^2} = \frac{2a^2}{r^4}\,\frac{(a(1-\cos\theta))}{a}$$

$$\frac{1}{p^2} = \frac{2a}{r^4}\,r$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$\boxed{p^2 = \frac{r^3}{2a}}$$

Pbm - 2

Form the ~~para~~ polar eqn of the parabola S.T $p^2 = ar$

Soln:

Polar eqn of the parabola is

$$\frac{2a}{r} = 1 - \cos\theta$$

$$\frac{1}{r} = \frac{1-\cos\theta}{2a} \longrightarrow (1)$$

Diff w.r.to θ, we get

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{2a}\sin\theta$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-r^2\sin\theta}{2a}$$

The p-r equation of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{-r^2\sin\theta}{2a}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{r^4\sin^2\theta}{4a^2}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{\sin^2\theta}{4a^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{(1-\cos\theta)^2}{4a^2} + \frac{\sin^2\theta}{4a^2} \quad [\because \text{by } (1)]$$

$$\Rightarrow \frac{1}{p^2} = \frac{(1-\cos\theta)^2 + \sin^2\theta}{4a^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{4a^2}\left[1+\cos^2\theta - 2\cos\theta + \sin^2\theta\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{4a^2}\left[2-2\cos\theta\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{4a^2}\,2(1-\cos\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{4a^2}\,2\left(\frac{2a}{r}\right) \quad [\because \text{by } (1)]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{ar}$$

$$\Rightarrow \boxed{p^2 = ar}$$

Pbm - 2

Find the Pedal equation of the curve $r^m = a^m \sin m\theta$

Solution:

Gn curve : $r^m = a^m \sin m\theta$ $\quad \sin m\theta = \frac{r^m}{a^m}$ $\rightarrow (1)$

Taking log on b.s, we get

$$m\log r = \log(a^m \sin m\theta)$$

$$\Rightarrow m\log r = \log a^m + \log\sin m\theta$$

$$\Rightarrow m\log r = m\log a + \log\sin m\theta$$

Diff w.r to θ, we get,

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{1}{\sin m\theta}(\cos m\theta).m$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \cot m\theta$$

$$\frac{dr}{d\theta} = r\cot m\theta$$

The $p-r$ equation of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(r^2\cot^2 m\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\cot^2 m\theta$$

$$\Rightarrow \frac{1}{p^2} = \frac{1+\cot^2 m\theta}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{\operatorname{cosec}^2 m\theta}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \cancel{\operatorname{cosec}^2 m\theta}\ \frac{1}{r^2\sin^2 m\theta} \times \frac{(a^m)^2}{(a^m)^2}$$

$$= \cancel{\frac{1}{p^2}}\ \frac{1}{r^2\sin^2 m\theta} \times \left(\frac{a^m}{r^m}\right)^2 \quad [\because \text{by } ①]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^m}{r^m}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^{2m}}{r^{2+2m}} = \left(\frac{a^m}{r^{m+1}}\right)^2$$

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$$\Rightarrow \frac{1}{p} = \frac{a^m}{r^{m+1}}$$

$$\Rightarrow \boxed{p = \frac{r^{m+1}}{a^m}}$$

Pbm - 4

Find the p-r equation of the curve

$r = \frac{a}{2}(1-\cos\theta)$

Soln:

Gn curve ; $r = \frac{a}{2}(1-\cos\theta)$

Diff. w.r.to θ, we get

$$\frac{dr}{d\theta} = \frac{a}{2}\sin\theta$$

The pr equ of the curve, be

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{a^2}{4}\sin^2\theta\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{4r^2 + a^2\sin^2\theta}{4r^4}$$

$$\Rightarrow \frac{1}{p^2} = \frac{4\left(\frac{a^2}{4}(1-\cos\theta)^2\right) + a^2\sin^2\theta}{4r^4}$$

$$= \frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{4r^4}$$

$$= \frac{a^2(1-\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta}{4r^4}$$

$$= \frac{a^2 + a^2\cos^2\theta + a^2\sin^2\theta - 2a^2\cos\theta}{4r^4}$$

$$= \frac{2a^2 - 2a^2\cos\theta}{4r^4}$$

$$= \frac{2a^2(1-\cos\theta)}{4r^4}$$

$$\frac{2a \cdot a(1-\cos\theta)}{2 \times \frac{1}{1/2} r^4}$$

$$= \frac{2a \cdot \frac{a}{2}(1-\cos\theta)}{2r^4}$$

$$\frac{2a \cdot \frac{a}{2}(1-\cos\theta)}{2r^4}$$

$$= \frac{2ar}{2r^4} = \frac{a}{r^3}$$

$$\Rightarrow \frac{1}{p^2} = \frac{a}{r^3}$$

$$\Rightarrow p^2 = \frac{r^3}{a}$$

$$\Rightarrow \boxed{ap^2 = r^3}$$

Ex. **Pbm:-5**

Find the $p-r$ eqn of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln:

Gn curve : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

W.k.T

The Parametric eqn of the ellipse is $x = a\cos\theta$, $y = b\sin\theta$

and the equation of the tangent at the point $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$\Rightarrow bx\cos\theta + ay\sin\theta = ab$$

$$\Rightarrow bx\cos\theta + ay\sin\theta - ab = 0$$

$$ax + by + c = 0$$

The length of the tangent p from origin is

$$p = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$ax + by + c = 0$$

$$p = \frac{-c}{\sqrt{a^2+b^2}}$$

$$p^2 = \frac{(ab)^2}{\left(\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}\right)^2}$$

$$p^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\frac{1}{p^2} = \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2b^2} \longrightarrow (1)$$

Also W.K.T the mutual relation between the cartesian co-ordinates and polar co-ordinates as $r^2 = x^2 + y^2$

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

$$\Rightarrow r^2 = a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta)$$

$$\Rightarrow r^2 = a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta$$

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$$\Rightarrow r^2 = a^2+b^2 - (a^2\sin^2\theta + b^2\cos^2\theta)$$

$$\Rightarrow r^2 = a^2+b^2 - \frac{a^2b^2}{p^2} \quad [\because \text{by } (7)]$$

$$\Rightarrow r^2-a^2-b^2 = \frac{-a^2b^2}{p^2}$$

$$\Rightarrow a^2+b^2-r^2 = \frac{a^2b^2}{p^2}$$

$$\Rightarrow \frac{a^2+b^2-r^2}{a^2b^2} = \frac{1}{p^2}$$

$$\Rightarrow \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2} - \frac{r^2}{a^2b^2} = \frac{1}{p^2}$$

$$\Rightarrow \boxed{\frac{1}{b^2} + \frac{1}{a^2} - \frac{r^2}{a^2b^2} = \frac{1}{p^2}}$$

Pbm - 6

Find the p-r eqn of the conic $\frac{l}{r} = 1+e\cos\theta$.

Soln:

Gn curve:

$$\frac{l}{r} = 1+e\cos\theta$$

$$\cancel{\frac{l}{r} = 1+e\cos}$$

$$\frac{1}{r} = \frac{1+e\cos\theta}{l}$$

$$r = \frac{l}{1+e\cos\theta}$$

$$\frac{dr}{d\theta} = \frac{-l}{(1+e\cos\theta)^2}\,(e(-\sin\theta))$$

$$\frac{dr}{d\theta} = \frac{el\sin\theta}{(1+e\cos\theta)^2}$$

The $p-r$ eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{e^2 l^2 \sin^2\theta}{(1+e\cos\theta)^4}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{e^2 l^2 \sin^2\theta}{\left(\frac{l^4}{r^4}\right)}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{\cancel{r^4}} \times \frac{\cancel{r^4}}{l^4}\,(e^2 l^2 \sin^2\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2\sin^2\theta}{l^2} \rightarrow (1)$$

Now

$$1+e\cos\theta = \frac{l}{r}$$

$$e\cos\theta = \frac{l}{r} - 1$$

$$e^2\cos^2\theta = \left(\frac{l}{r} - 1\right)^2$$

$$e^2(1-\sin^2\theta) = \left(\frac{l}{r}-1\right)^2$$

$$e^2 - e^2\sin^2\theta = \left(\frac{l}{r}-1\right)^2$$

$$e^2 - \left(\frac{l}{r}-1\right)^2 = e^2\sin^2\theta \rightarrow (2)$$

Substitute (2) in (1) we, get.

$$(1) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left(e^2 - \left(\frac{l}{r}-1\right)^2\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left(e^2 - \left(\frac{l^2}{r^2}+1-2\frac{l}{r}\right)\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left(\frac{e^2r^2 - l^2 - r^2 + 2lr}{r^2}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2l^2}\left(e^2r^2 - l^2 - r^2 + 2lr\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{l^2 + e^2r^2 - l^2 - r^2 + 2lr}{l^2r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{e^2r^2 - r^2 + 2lr}{l^2r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{r\left[e^2r - r + 2l\right]}{l^2r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{e^2r - r + 2l}{rl^2}$$

$$p^2 = \frac{r d^2}{e^2 - r + a^2}$$

Pbm - 7

Find the $p-r$ eqn of the curve

$r = a e^{\theta \cot \alpha}$

Soln:

Given curve : $r = a e^{\theta \cot \alpha}$

$$\frac{dr}{d\theta} = a e^{\theta \cot \alpha} \cdot (\cot \alpha)$$

$$\frac{dr}{d\theta} = r \cot \alpha$$

The $p-r$ eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(r^2 \cot^2 \alpha)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2} \cot^2 \alpha$$

$$\Rightarrow \frac{1}{p^2} = \frac{1 + \cot^2 \alpha}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{\operatorname{cosec}^2 \alpha}{r^2}$$

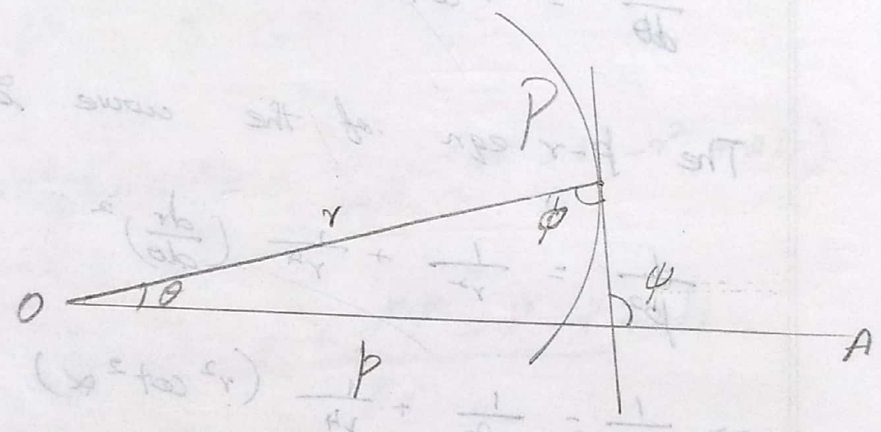
$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \alpha}$$

$$\Rightarrow \quad p^2 = r^2 \sin^2\alpha$$

$$\Rightarrow \quad \boxed{p = r \sin \alpha}$$

Remark:

For some curves, it is not easy to calculate the radius of curvature from their polar co-ordinates. In those cases, we can use the following formula.



$$\sin\phi = r\frac{d\theta}{ds}; \quad \tan\phi = r\frac{d\theta}{dr}$$

$$\cos\phi = \frac{dr}{ds}; \quad p = r\sin\phi$$

$$\frac{dp}{dr} = r\cos\phi\,\frac{d\phi}{dr} + \sin\phi$$

$$= r\frac{dr}{ds}\frac{d\phi}{dr} + r\frac{d\theta}{ds}$$

$$= r\frac{d\phi}{ds} + r\frac{d\theta}{ds}$$

$$= r\frac{d}{ds}(\theta+\phi)$$

$$\frac{dp}{dr} = r\frac{d\psi}{ds}$$

$$\Rightarrow \frac{1}{r}\frac{dp}{dr} = \frac{d\psi}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = r\frac{dr}{dp}$$

$$\Rightarrow \boxed{\rho = r\frac{dr}{dp}}$$

Pbm-8

Find the radius of curvature of the Cardiod $r = a(1-\cos\theta)$

Prove that $\frac{\rho^2}{r}$ is constant.

Soln:

In Pbm-1 we shown that the p-r eqn of the gn curve is

$$p^2 = \frac{r^3}{2a}$$

$$\Rightarrow \boxed{2ap^2 = r^3}$$

Diff w.r.t p, we get

$$4ap = 3r^2 \frac{dr}{dp}$$

$$4ap = 3r \cdot r \frac{dr}{dp}$$

$$\frac{4ap}{3r} = r\frac{dr}{dp}$$

$$\rho = \frac{4ap}{3r}$$

$$\rho = \frac{4a}{3r}\left(\frac{r^3}{2a}\right)^{1/2}$$

$$\rho = \frac{4a}{3r} \frac{r^{3/2}}{\sqrt{2}\sqrt{a}}$$

$$\rho = \frac{\cancel{2}\sqrt{2}\sqrt{\cancel{2}}\sqrt{a}\sqrt{a}}{3\cancel{r}} \frac{r\sqrt{r}}{\sqrt{\cancel{2}}\sqrt{a}}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

$$\rho^2 = \frac{4}{9}2ar$$

$$\frac{\rho^2}{r} = \frac{4}{9}2a$$

$$\frac{\rho^2}{r} = \frac{8a}{9} = \text{constant}$$

Pbm - 9

Find the radius of curvature of the curve $r^2 = a^2\sin 2\theta$

Soln:

Gin curve : $r^2 = a^2\sin 2\theta$

Diff w.r.t θ, we get.

$$2r \frac{dr}{d\theta} = a^2 \cos 2\theta \,(2)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\cancel{2}a^2 \cos 2\theta}{\cancel{2}r}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{a^2 \cos 2\theta}{r}$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{a^4 \cos^2 2\theta}{r^2}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{r^4 + a^4 \cos^2 2\theta}{r^6}$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^4 \sin^2 2\theta + a^4 \cos^2 2\theta}{r^6}$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^4}{r^6}$$

$$\Rightarrow p^2 = \frac{r^6}{a^4}$$

$$\Rightarrow \boxed{p = \frac{r^3}{a^2}}$$

Diff w. r to r, we get

$$\frac{dp}{dr} = \frac{1}{a^2}(3r^2)$$

$$\Rightarrow \frac{dp}{dr} = \frac{3r^2}{a^2}$$

$\frac{dp}{dr} = \frac{3r}{a^2} \times r$

$$\Rightarrow \frac{1}{r}\frac{dp}{dr} = \frac{3r}{a^2}$$

$\frac{1}{r}\frac{dp}{dr} = \frac{3r}{a^2}$

$$\Rightarrow r\frac{dr}{dp} = \frac{a^2}{3r}$$

$$\Rightarrow \rho = \frac{a^2}{3r}$$

Pbm - 10.

Find the pedal eqn of the curve $x^2+y^2 = 2ax$ and deduce its radius of curvature.

Soln:

Obviously the gn eqn represents the eqn of the circle.

Put $x = r\cos\theta$ &

$y = r\sin\theta$

$\therefore r^2 = 2ar\cos\theta$

$\Rightarrow r = 2a\cos\theta$ is the polar eqn of the circle.

Now $\frac{dr}{d\theta} = -2a\sin\theta$

The p-r eqn of the gn curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(4a^2\sin^2\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{r^2 + 4a^2\sin^2\theta}{r^4}$$

$$\Rightarrow \frac{1}{p^2} = \frac{4a^2\cos^2\theta + 4a^2\sin^2\theta}{r^4}$$

$$\Rightarrow \frac{1}{p^2} = \frac{4a^2}{r^4}$$

$$\Rightarrow p^2 = \frac{r^4}{4a^2}$$

$$\boxed{p = \frac{r^2}{2a}}$$

Diff w.r.t. r, we get

$$\frac{dp}{dr} = \frac{1}{2a}(2r)$$

$$\frac{1}{r}\frac{dp}{dr} = \frac{1}{a}$$

$$r\frac{dr}{dp} = a$$

$$\boxed{\rho = a}$$

Pbm - 11

Find the pedal eqn of the curve $r^n = a^n \sin n\theta$. Hence find the radius of curvature.

Soln:

In Pbm - 3, we shown that the p-r eqn of the gn curve is

$$\boxed{p = \frac{r^{n+1}}{a^n}}$$

Diff w.r to r, we get

$$\frac{dp}{dr} = \frac{1}{a^n}(n+1) r^{n+1-1}$$

$$\Rightarrow \frac{dp}{dr} = \frac{1}{a^n}(n+1) r^n$$

$$= \cancel{\frac{1}{a^n}(n+1) a^n \sin n\theta}$$

$$\Rightarrow \frac{1}{r}\frac{dp}{dr} = \frac{(n+1) r^n}{a^n} \frac{1}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dp}{dr} = \frac{(n+1) r^{n-1}}{a^n}$$

$$r \frac{dr}{dp} = \frac{a^n}{(n+1)\, r^{n-1}}$$

$$\boxed{\rho = \frac{a^n\, r^{1-n}}{n+1}}$$

Pbm - 12

find the radius of curvature for the general conic

$$\frac{l}{r} = 1 + e\cos\theta$$

Soln.

In pbm-6, we shown that the p-r eqn of the gn curve is

$$\boxed{p^2 = \frac{l^2 r}{e^2 r - r + 2l}}$$

Diff w.r.to r, we get

$$2p\frac{dp}{dr} = l^2\left[\frac{(e^2 r - r + 2l) - r(e^2 - 1)}{(e^2 r - r + 2l)^2}\right]$$

$$\Rightarrow 2p\frac{dp}{dr} = l^2\left[\frac{e^2 r - r + 2l - e^2 r + r}{(e^2 r - r + 2l)^2}\right]$$

$$2p\frac{dp}{dr} = l^2\left[\frac{2l}{(e^2 r - r + 2l)^2}\right]$$

$$\Rightarrow \quad p\,\frac{dp}{dr} = \frac{l^3}{(e^2r - r + 2l)^2}$$

$$\Rightarrow \quad \frac{dp}{dr} = \frac{1}{p}\,\frac{l^3}{(e^2r - r + 2l)^2}$$

$$\Rightarrow \quad \frac{dr}{dp} = \frac{p(e^2r - r + 2l)^2}{l^3}$$

$$\Rightarrow \quad r\,\frac{dr}{dp} = \frac{pr(e^2r - r + 2l)^2}{l^3}$$

$$\Rightarrow \quad \rho = \frac{r(e^2r - r + 2l)^2}{l^3}\left(\frac{l^2 r}{e^2r - r + 2l}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \rho = \frac{r^{3/2}}{l^2}\,(e^2r - r + 2l)^{2 - \frac{1}{2}}$$

$$\Rightarrow \quad \rho = \frac{r^{3/2}}{l^2}\,(e^2r - r + 2l)^{3/2}$$

$$\Rightarrow \quad \boxed{\rho = \frac{(e^2r^2 - r^2 + 2lr)^{3/2}}{l^2}}$$

H.W

1. Find the p-r eqn $r\sin\theta + a = 0$

Soln:

Gn Curve : $r\sin\theta + a = 0$

Diff w.r to θ, we get

$$r\cos\theta + \sin\theta \frac{dr}{d\theta} = 0$$

$$\sin\theta \frac{dr}{d\theta} = -r\cos\theta$$

$$\frac{dr}{d\theta} = \frac{-r\cos\theta}{\sin\theta} = -r\cot\theta$$

$$\frac{dr}{d\theta} = -r\cot\theta$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(r^2\cot^2\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{\cot^2\theta}{r^2}$$

$$\frac{1}{p^2} = \frac{1+\cot^2\theta}{r^2} = \frac{\operatorname{cosec}^2\theta}{r^2}$$

$$\frac{1}{p^2} = \frac{1}{\sin^2\theta\, r^2}$$

$$p^2 = \sin^2\theta\, r^2$$

$$p^2 = \sin^2\theta\, \frac{a^2}{\sin^2\theta}$$

$$p^2 = a^2$$

$$p = \pm a$$

$$\boxed{p + a = 0}$$

H.W

2) Find the p-r eqn $r = a\sin\theta$

Soln:

Gn Curve $r = a\sin\theta$

~~da~~ Diff w r to θ we get

$$\frac{dr}{d\theta} = a\cos\theta$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} a^2\cos^2\theta$$

$$\frac{1}{p^2} = \frac{r^2 + a^2\cos^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2\sin^2\theta + a^2\cos^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2}{r^4}$$

$$p^2 = \frac{r^4}{a^2}$$

$$p = \frac{r^2}{a}$$

$$\boxed{pa = r^2}$$

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4) Find the p-r eqn $r^m = a^m \cos m\theta$

H.W

Solution:

Gn Curve, $r^m = a^m \cos m\theta$

Taking log on both sides, we get,

$\log r^m = \log (a^m \cos m\theta)$

$m \log r = \log a^m + \log \cos m\theta$

$m \log r = m \log a + \log \cos m\theta$

Diff w.r to θ, we get,

$m \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\cos m\theta} (-\sin m\theta)(m)$

$\frac{1}{r} \frac{dr}{d\theta} = -\tan m\theta$

$\frac{dr}{d\theta} = -r \tan m\theta$

The p-r eqn of the curve is

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2$

$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} (r^2 \tan^2 m\theta)$

$\frac{1}{p^2} = \frac{1 + \tan^2 m\theta}{r^2}$

$\frac{1}{p^2} = \frac{\sec^2 m\theta}{r^2}$

$$\frac{1}{p^2} = \frac{1}{\cos^2 m\theta \; r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} \frac{1}{\left(\frac{r^m}{a^m}\right)^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^m}{r^m}\right)^2$$

$$\frac{1}{p^2} = \frac{a^{2m}}{r^{2+2m}}$$

$$\frac{1}{p^2} = \frac{a^{2m}}{r^{2(m+1)}}$$

$$\frac{1}{p^2} = \left(\frac{a^m}{r^{(m+1)}}\right)^2$$

$$\frac{1}{p} = \frac{a^m}{r^{(m+1)}}$$

$$p = \frac{r^{(m+1)}}{a^m}$$

$$\boxed{p a^m = r^{(m+1)}}$$

---

6) Find the $p-r$ eqn $r^2 \cos 2\theta = a^2$

H.W

Solution:

Gn Curve $r^2 \cos 2\theta = a^2$

Diff w.r to θ, we get,

$$r^2(-\sin 2\theta)(2) + \cos 2\theta \left(2r \frac{dr}{d\theta}\right) = 0$$

$$2r \cos 2\theta \frac{dr}{d\theta} = 2r^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{2r^2 \sin 2\theta}{2r \cos 2\theta}$$

$$\frac{dr}{d\theta} = r \tan 2\theta$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(r^2 \tan^2 2\theta\right)$$

$$\frac{1}{p^2} = \frac{1 + \tan^2 2\theta}{r^2}$$

$$\frac{1}{p^2} = \frac{\sec^2 2\theta}{r^2}.$$

$$\frac{1}{p^2} = \frac{1}{\cos^2 2\theta} \; \frac{1}{r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} \; \frac{1}{\left(\frac{a^2}{r^2}\right)^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} \; \frac{r^4}{a^4}$$

$$\frac{1}{p^2} = \frac{r^2}{a^4}$$

$$\frac{1}{p} = \frac{r}{a^2}.$$

$$p = \frac{a^2}{r}$$

$$\boxed{pr = a^2}$$

H.W

7) Find the $p-r$ eqn of the curve

$r\theta = a$

Soln:

Gn curve $r\theta = a$

Diff w.r to θ, we get

$$r + \theta \frac{dr}{d\theta} = 0$$

$$\theta \frac{dr}{d\theta} = -r$$

$$\frac{dr}{d\theta} = \frac{-r}{\theta}$$

The $p-r$ eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{r^2}{\theta^2}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\frac{1}{\theta^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\frac{1}{a^2/r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\frac{r^2}{a^2}$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{a^2}}$$

Evolute and Involute

Defn: Evolute

The locus of center of curvature for a curve is called the evolute of the curve.

Pbm - 13

S.T in the parabola $y^2 = 4ax$ at the pt t,

$$\rho = -2a(1+t^2)^{3/2}$$

$$x = 2a + 3at^2$$

$$y = -2at^3$$

. Deduce the eqn of evolute.

Soln:

In Pbm - 11 of Unit I ; we shown that

$$\rho = -2a(1+t^2)^{3/2}$$

$$x = 2a + 3at^2 \rightarrow (1)$$

$$y = -2at^3 \rightarrow (2)$$

Eliminating t from x and y

$$(1) \Rightarrow x = 2a + 3at^2$$

$$\Rightarrow x - 2a = 3at^2$$

$$\Rightarrow \frac{x - 2a}{3a} = t^2$$

$$\Rightarrow \left(\frac{x-2a}{3a}\right)^{3/2} = (t^2)^{3/2} = t^3$$

$$② \Rightarrow y = 2a\left(\frac{x-2a}{3a}\right)^{3/2}$$

$$y^2 = 4a^2\left(\frac{x-2a}{3a}\right)^3$$

$$y^2 = \frac{4}{27a}(x-2a)^3$$

$$27ay^2 = 4(x-2a)^3$$

The Locus of (x,y) is

$$27ay^2 = 4(x-2a)^3$$

Pbm - 14

U.Q. (8m) Find the evolute of the ellipse

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

Soln:

W.K.T The parametric eqn of the ellipse is

$$x = a\cos\theta, \quad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= -\frac{b}{a}\cot\theta$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{-b}{a}\cot\theta\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{-b}{a}(-\operatorname{cosec}^2\theta) \times \frac{1}{-a\sin\theta}$$

$$= \frac{-b}{a^2\sin^3\theta}$$

Consider,

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow X = a\cos\theta - \frac{-\frac{b}{a}(\cot\theta)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2\sin^3\theta}}$$

$$\Rightarrow X = a\cos\theta - \frac{(\cot\theta)\left(\frac{a^2+b^2\cot^2\theta}{a^2}\right)}{\frac{1}{a\sin^3\theta}}$$

$$\Rightarrow X = a\cos\theta - \frac{\frac{\cos\theta\,(a^2+b^2\cot^2\theta)}{\sin\theta \times a^2}}{\frac{1}{a\sin^3\theta}}$$

$$\Rightarrow X = a\cos\theta - \frac{\cos\theta\,\sin^2\theta\,(a^2+b^2\cot^2\theta)}{a}$$

$$\Rightarrow x = \frac{a^2\cos\theta - a^2\cos\theta\sin^2\theta - b^2\cot^2\theta\cos\theta\,\sin^2\theta}{a}$$

$$\Rightarrow x = \frac{a\cos\theta(1-\sin^2\theta) - b^2\frac{\cos^2\theta}{\sin^2\theta}\cos\theta\sin^2\theta}{a}$$

$$\Rightarrow x = \frac{a^2\cos^3\theta - b^2\cos^3\theta}{a}$$

$$\Rightarrow x = \frac{(a^2-b^2)\cos^3\theta}{a} \longrightarrow (1)$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$\Rightarrow Y = b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{\frac{-b}{a^2\sin^3\theta}}$$

$$\Rightarrow Y = b\sin\theta + \frac{(a^2+b^2\cot^2\theta)}{\frac{-a^2b}{a\sin^3\theta}}$$

$$\Rightarrow Y = b\sin\theta - \frac{\sin^3\theta(a^2+b^2\cot^2\theta)}{b}$$

$$\Rightarrow Y = b\sin\theta - \frac{a^2\sin^3\theta - b^2\cot^2\theta\sin^3\theta}{b}$$

$$\Rightarrow Y = \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\frac{\cos^2\theta}{\sin^2\theta}\sin^3\theta}{b}$$

$$\Rightarrow y = \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\cos^2\theta\sin\theta}{b}$$

$$\Rightarrow y = \frac{b^2\sin\theta(1-\cos^2\theta) - a^2\sin^3\theta}{b}$$

$$\Rightarrow y = \frac{b^2\sin^3\theta - a^2\sin^3\theta}{b}$$

$$\Rightarrow y = \frac{(b^2-a^2)\sin^3\theta}{b}$$

$$\Rightarrow y = \frac{-(a^2-b^2)\sin^3\theta}{b} \rightarrow (2)$$

From (1) and (2)

$$\cos\theta = \left[\frac{x}{\left(\frac{a^2-b^2}{a}\right)}\right]^{1/3} \quad \&$$

$$\sin\theta = \left[\frac{-y}{\left(\frac{a^2-b^2}{b}\right)}\right]^{1/3}$$

$$\Rightarrow \cos\theta = \left(\frac{ax}{a^2-b^2}\right)^{1/3} \quad \&$$

$$\sin\theta = \left(\frac{-by}{a^2-b^2}\right)^{1/3}$$

To eliminate θ,

squaring and adding we get.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{ax}{a^2-b^2}\right)^{2/3} + \left(\frac{-by}{a^2-b^2}\right)^{2/3} = 1$$

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$

The Locus of (x, y) is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$

Pbm - 15

S.T the evolute of a cycloid $x = a(\theta - \sin\theta)$ $y = a(1-\cos\theta)$ is another cycloid.

Soln:

Gn curve $x = a(\theta - \sin\theta)$

$y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta) \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(\cot \theta/2)\ \frac{d\theta}{dx}$$

$$= (-\operatorname{cosec}^2 \tfrac{\theta}{2})\tfrac{1}{2}\left(\frac{1}{a(1-\cos\theta)}\right)$$

$$= \frac{-\operatorname{cosec}^2 \theta/2}{2a(1-\cos\theta)}$$

$$= \frac{-1}{2a(2\sin^2\theta/2)(\sin^2\theta/2)}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{4a\sin^4\theta/2}$$

Now

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow X = a(\theta - \sin\theta) - \frac{(\cot \theta/2)(1+\cot^2\theta/2)}{\frac{-1}{4a\sin^4\theta/2}}$$

$$\Rightarrow X = a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2} \times \operatorname{cosec}^2\theta/2 \times 4a\sin^4\theta/2$$

$$\Rightarrow X = a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2} \times \frac{1}{\sin^2\theta/2} \times 4a\sin^4\theta/2$$

$$\Rightarrow X = a(\theta - \sin\theta) + 4a\cos\theta/2\ \sin\theta/2$$

$\Rightarrow x = a(\theta - \sin\theta) + 2a\sin\theta$

$\Rightarrow x = a\theta - a\sin\theta + 2a\sin\theta$

$\Rightarrow x = a\theta + a\sin\theta$

$\Rightarrow x = a(\theta + \sin\theta)$

Also

$$Y = y + \frac{(1 + y_1^2)}{y_2}$$

$$\Rightarrow Y = a(1-\cos\theta) + \frac{(1+\cot^2\theta/2)}{\frac{-1}{4a\sin^4\theta/2}}$$

$\Rightarrow Y = a(1-\cos\theta) - \operatorname{cosec}^2\theta/2 \times 4a\sin^4\theta/2$

$$\Rightarrow Y = a(1-\cos\theta) - \frac{1}{\sin^2\theta/2} \times 4a\sin^4\theta/2$$

$\Rightarrow Y = a(1-\cos\theta) - 4a\sin^2\theta/2$

$\Rightarrow Y = a(1-\cos\theta) - 2a(2\sin^2\theta/2)$

$\Rightarrow Y = a(1-\cos\theta) - 2a(1 - \cos\theta)$

$\Rightarrow Y = a(1-\cos\theta)(1-2)$

$\Rightarrow Y = -a(1-\cos\theta)$

The Locus of (x, y)

$x = a(\theta + \sin\theta)$, $y = -a(1 - \cos\theta)$

This is also a cycloid.

Pbm - 16

S.T the evolute of the hyperbola

$\frac{x^2}{a} - \frac{y^2}{b} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$

Soln :

W.K.T the parametric eqn of the hyperbola $x = a\sec\theta$, $y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ $\quad \frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$$= \frac{b}{a}\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$= \frac{b}{a}\times\frac{1}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{b}{a}\operatorname{cosec}\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right)\frac{d\theta}{dx}$$

$$= \frac{-b}{a}\operatorname{cosec}\theta\cot\theta \times \frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{-b/a \ \frac{1}{\sin\theta} \times \frac{\cos\theta}{\sin\theta}}{a \ \frac{1}{\cos\theta} \ \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{-b}{a}\frac{\cos\theta}{\sin^2\theta} \times \frac{\cos^2\theta}{a\sin\theta}$$

$$= \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2}\cot^3\theta$$

Now

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow X = a\sec\theta - \frac{\frac{b}{a}\operatorname{cosec}\theta\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{-\frac{b}{a^2}\cot^3\theta}$$

$$\Rightarrow X = a\sec\theta + \frac{\operatorname{cosec}\theta\left(\frac{a^2+b^2\operatorname{cosec}^2\theta}{a^2}\right)}{\frac{1}{a}\cot^3\theta}$$

$$\Rightarrow X = a\sec\theta + a\operatorname{cosec}\theta\tan^3\theta\,\frac{(a^2+b^2\operatorname{cosec}^2\theta)}{a^2}$$

$$\Rightarrow X = a\sec\theta + a\frac{1}{\sin\theta}\,\frac{\sin^3\theta}{\cos^3\theta}\,\frac{(a^2+b^2\operatorname{cosec}^2\theta)}{a^2}$$

$$\Rightarrow X = a\sec\theta + \frac{\sin^2\theta}{\cos^3\theta}\,\frac{(a^2+b^2\operatorname{cosec}^2\theta)}{a}$$

$$\Rightarrow X = \frac{a^2\sec\theta + a^2\frac{\sin^2\theta}{\cos^3\theta} + b^2\operatorname{cosec}^2\theta\frac{\sin^2\theta}{\cos^3\theta}}{a}$$

$$\Rightarrow X = \frac{a^2\sec\theta + a^2\sin^2\theta\sec^3\theta + b^2\sec^3\theta}{a}$$

$$\Rightarrow X = \frac{a^2\sec\theta(1+\sin^2\theta\sec^2\theta) + b^2\sec^3\theta}{a}$$

$$\Rightarrow X = \frac{a^2\sec\theta\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right) + b^2\sec^3\theta}{a}$$

$$\Rightarrow x = \frac{a^2 \sec\theta(1+\tan^2\theta) + b^2\sec^3\theta}{a}$$

$$\Rightarrow x = \frac{a^2\sec^3\theta + b^2\sec^3\theta}{a}$$

$$\Rightarrow x = \frac{(a^2+b^2)(\sec^3\theta)}{a} \longrightarrow (1)$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$\Rightarrow Y = b\tan\theta + \frac{\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{-\frac{b}{a^2}\cot^3\theta}$$

$$\Rightarrow y = b\tan\theta - \frac{(a^2+b^2\operatorname{cosec}^2\theta)}{a^2} \times \frac{a^2\tan^3\theta}{b}$$

$$\Rightarrow Y = b\tan\theta - \frac{(a^2\tan^3\theta + b^2\operatorname{cosec}^2\theta\tan^3\theta)}{b}$$

$$\Rightarrow Y = \frac{b^2\tan\theta - a^2\tan^3\theta - b^2\frac{1}{\sin^2\theta}\frac{\sin^3\theta}{\cos^3\theta}}{b}$$

$$\Rightarrow Y = \frac{b^2\tan\theta - a^2\tan^3\theta - b^2\frac{\sin\theta}{\cos\theta}\frac{1}{\cos^2\theta}}{b}$$

$$\Rightarrow Y = \frac{b^2\tan\theta - a^2\tan^3\theta + b^2\tan\theta\sec^2\theta}{b}$$

$$\Rightarrow y = \frac{b^2 \tan\theta(1-\sec^2\theta) - a^2\tan^3\theta}{b}$$

$$\Rightarrow y = \frac{-b^2\tan^3\theta - a^2\tan^3\theta}{b}$$

$$\Rightarrow y = -\frac{(a^2+b^2)}{b}\tan^3\theta \longrightarrow (2)$$

From (1) and (2)

$$\sec\theta = \left(\frac{x}{\frac{(a^2+b^2)}{a}}\right)^{1/3}$$

$$\sec\theta = \left(\frac{ax}{a^2+b^2}\right)^{1/3}$$

$$\tan\theta = \left(\frac{-y}{\frac{(a^2+b^2)}{b}}\right)^{1/3}$$

$$\tan\theta = \left(\frac{-by}{a^2+b^2}\right)^{1/3}$$

To eliminate θ, using the Identify $\sec^2\theta - \tan^2\theta = 1$

we get

$$\left(\frac{ax}{a^2+b^2}\right)^{2/3} - \left(\frac{-by}{a^2+b^2}\right)^{2/3} = 1$$

$$\frac{(ax)^{2/3}}{(a^2+b^2)^{2/3}} - \frac{(by)^{2/3}}{(a^2+b^2)^{2/3}} = 1$$

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

The Locus of (x, y) is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

Pbm - 17

S.T the eqn of the evolute of the curve $2xy = a^2$ is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$$

Soln:

Gn Curve $2xy = a^2$

Diff w.r to x, we get

$$2\left(x\frac{dy}{dx} + y\right) = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{d^2y}{dx^2} = \frac{(-x)\frac{dy}{dx} + y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-x)\left(-\frac{y}{x}\right) + y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2}$$

Now,

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow X = x + \frac{(y/x)\left(1+ \frac{y^2}{x^2}\right)}{2y/x^2}$$

$$\Rightarrow X = x + \frac{\left(\frac{x^2+y^2}{x^2}\right)}{2/x}$$

$$\Rightarrow X = x + \frac{x^2+y^2}{2x}$$

$$\Rightarrow X = \frac{2x^2 + x^2 + y^2}{2x}$$

$$\Rightarrow X = \frac{3x^2+y^2}{2x}$$

Also $Y = y + \frac{(1+y_1^2)}{y_2}$

$$\Rightarrow Y = y + \frac{\left(1 + y^2/x^2\right)}{2y/x^2}$$

$$\Rightarrow Y = y + \left(\frac{x^2+y^2}{x^2}\right)\left(\frac{x^2}{2y}\right)$$

$$\Rightarrow Y = y + \frac{x^2+y^2}{2y}$$

$$\Rightarrow Y = \frac{2y^2+x^2+y^2}{2y}$$

$$\Rightarrow Y = \frac{x^2+3y^2}{2y}$$

$$X + Y = \frac{3x^2+y^2}{2x} + \frac{x^2+3y^2}{2y}$$

$$= \frac{y(3x^2+y^2) + x(x^2+3y^2)}{2xy}$$

$$= \frac{3x^2y+y^3+x^3+3xy^2}{2xy}$$

$$X+Y = \frac{(x+y)^3}{2xy} = \frac{(x+y)^3}{a^2}$$

consider

$$\frac{(X+Y)^{2/3}}{a^{2/3}} = \left(\frac{(x+y)^3}{a^2}\right)^{2/3} \times \frac{1}{a^{2/3}}$$

$$= \frac{(x+y)^2}{(a^2)^{2/3}} \cdot \frac{1}{a^{2/3}}$$

$$= \frac{(x+y)^2}{a^{4/3+2/3}} = \frac{(x+y)^2}{a^2}$$

Also

$$X - Y = \frac{3x^2+y^2}{2x} - \frac{x^2+3y^2}{2y}$$

$$= \frac{y(3x^2+y^2) - x(x^2+3y^2)}{2xy}$$

$$= \frac{3x^2y + y^3 - x^3 - 3xy^2}{a^2}$$

$$= \frac{-(x^3 - y^3 - 3x^2y + 3xy^2)}{a^2} = \frac{-(x-y)^3}{a^2}$$

$$\frac{(X-Y)^{2/3}}{a^{2/3}} = \left(-\frac{(x-y)^3}{a^2}\right)^{2/3} \cdot \frac{1}{a^{2/3}}$$

$$= \frac{(x-y)^2}{a^{4/3+2/3}} = \frac{(x-y)^2}{a^2}$$

Consider

$$\frac{(X+Y)^{2/3}}{a^{2/3}} - \frac{(X-Y)^{2/3}}{a^{2/3}} = \frac{(x+y)^2}{a^2} - \frac{(x-y)^2}{a^2}$$

$$= \frac{1}{a^2}\left[x^2+y^2+2xy - x^2 - y^2 + 2xy\right] = \frac{1}{a^2}(4xy)$$

$$= \frac{1}{a^2}(2(2xy)) = \frac{1}{a^2} 2a^2 = 2$$

$$\frac{(X+Y)^{2/3}}{a^{2/3}} - \frac{(X-Y)^{2/3}}{a^{2/3}} = 2$$

$$(X+Y)^{2/3} - (X-Y)^{2/3} = 2a^{2/3}$$

The locus of (X, Y) is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$$

H.W 5) $r = a\sin^3 \theta/3$ Find the p-r eqn.

Soln:

Gn Curve,

$$r = a\sin^3 \theta/3$$

Diff w.r to r, we get

$$\frac{dr}{d\theta} = a\,3\sin^2\theta/3 \cos\theta/3 \times \frac{1}{3}$$

$$= a\sin^2\theta/3 \cos\theta/3$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(a^2\sin^4\theta/3 \cos^2\theta/3\right)$$

$$\frac{1}{p^2} = \frac{r^2 + a^2\sin^4\theta/3\cos^2\theta/3}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2\sin^6\theta/3 + a^2\sin^4\theta/3\cos^2\theta/3}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2\sin^4\theta/3\left(\sin^2\theta/3 + \cos^2\theta/3\right)}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2\sin^4\theta/3}{r^4}$$

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$$\frac{1}{p^2} = \frac{(a \sin \frac{3\theta}{3})(a \sin \theta/3)}{r^4}$$

$$\frac{1}{p^2} = \frac{r a \sin \theta/3}{r^4}$$

$$\frac{1}{p^2} = \frac{a \sin \theta/3}{r^3}$$

$$\frac{r^3}{p^2} = a\left(\frac{r}{a}\right)^{1/3}$$

$$= r^{1/3} a^{1-1/3}$$

$$\frac{r^3}{p^2} = r^{1/3} a^{2/3}$$

$$\frac{r^3}{r^{1/3}} = p^2 a^{2/3}$$

$$r^{3-1/3} = p^2 a^{2/3}$$

$$r^{8/3} = p^2 a^{2/3}$$

$$r^8 = p^6 a^2$$

$$r^4 = p^3 a$$

$$\boxed{p^3 a = r^4}$$

Hw(3) $r^2 \sin 2\theta + a^2 = 0$ Find the p-r eqn

Soln:

Gn Curve: $r^2 \sin 2\theta + a^2 = 0$

$$r^2 \cos 2\theta (2) + \sin 2\theta \left(2r \frac{dr}{d\theta}\right) = 0$$

$$\frac{dr}{d\theta} = \frac{-2r^2 \cos 2\theta}{2r \sin 2\theta}$$

$$\frac{dr}{d\theta} = -r \cot 2\theta$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(r^2 \cot^2 2\theta\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2} \cot^2 2\theta$$

$$\frac{1}{p^2} = \frac{1 + \cot^2 2\theta}{r^2}$$

$$\frac{1}{p^2} = \frac{\operatorname{cosec}^2 2\theta}{r^2}$$

$$\frac{1}{p^2} = \frac{1}{\sin^2 2\theta} \quad \frac{1}{r^2}$$

$$\frac{1}{p^2} = \frac{r^4}{a^4} \quad \frac{1}{r^2}$$

$$\frac{1}{p^2} = \frac{r^2}{a^4}$$

$$p^2 = \frac{a^4}{r^2}$$

$$p^2 r^2 = a^4$$

$$pr = \pm a^2$$

$$pr \pm a^2 = 0$$

Find the radius of curvature

H.W

2. $r \cos^2 \frac{\theta}{2} = a$

Soln:

Gin Curve

$$\frac{1}{2} r \, 2\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\right) + \cos^2\frac{\theta}{2}\frac{dr}{d\theta} = 0$$

$$\cos^2\frac{\theta}{2}\frac{dr}{d\theta} = r\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\frac{dr}{d\theta} = \frac{r\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$

$$\frac{dr}{d\theta} = r\tan\frac{\theta}{2}$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(r^2\tan^2\frac{\theta}{2}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{\tan^2\frac{\theta}{2}}{r^2}$$

$$\frac{1}{p^2} = \frac{1+\tan^2\frac{\theta}{2}}{r^2}$$

$$\frac{1}{p^2} = \frac{\sec^2\frac{\theta}{2}}{r^2}$$

$$\frac{1}{p^2} = \frac{r/a}{r^2}$$

$$= \frac{r}{ar^2}$$

$$\frac{1}{p^2} = \frac{1}{ar}$$

$$\boxed{p^2 = ar}$$

Diff w. r to p, we get

$$2p = a\frac{dr}{dp}$$

$$2\sqrt{a}\sqrt{r} = a\frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{2\sqrt{a}\sqrt{r}}{\sqrt{a}\sqrt{a}}$$

$$\frac{dr}{dp} = \frac{2\sqrt{r}}{\sqrt{a}}$$

$$\frac{dr}{dp} = 2\sec\theta/2$$

$$r\frac{dr}{dp} = 2r\sec\theta/2$$

$$\rho = 2\frac{a}{\cos^2\theta/2}\sec\theta/2$$

$$\rho = 2a\sec^2\theta/2\sec\theta/2$$

$$\boxed{\rho = 2a\sec^3\theta/2}$$

4) S.T the p-r eqn of the curve $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{3r}$

H.W

Soln.

Gin Curve

$$r^2 = a^2 \cos 2\theta$$

Diff w.r to θ, we get

$$2r \frac{dr}{d\theta} = a^2(-\sin 2\theta)(2)$$

$$\frac{dr}{d\theta} = \frac{-2a^2 \sin 2\theta}{2r}$$

$$\frac{dr}{d\theta} = \frac{-a^2}{r} \sin 2\theta$$

The p-r eqn of curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{a^4}{r^2} \sin^2 2\theta\right)$$

$$\frac{1}{p^2} = \frac{r^4 + a^4 \sin^2 2\theta}{r^6}$$

$$\frac{1}{p^2} = \frac{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta}{r^6}$$

$$\frac{1}{p^2} = \frac{a^4(\cos^2 2\theta + \sin^2 2\theta)}{r^6}$$

$$\frac{1}{p^2} = \frac{a^4}{r^6}$$

$$p^2 = \frac{r^6}{a^4}$$

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$$\boxed{p = \frac{r^3}{a^2}}$$

Diff w. r to r, we get.

$$\frac{dp}{dr} = \frac{1}{a^2} 3r^2$$

$$\frac{dp}{dr} = \frac{3r \cdot r}{a^2}$$

$$\frac{1}{r}\frac{dp}{dr} = \frac{3r}{a^2}$$

$$r\frac{dr}{dp} = \frac{a^2}{3r}$$

$$\boxed{\rho = \frac{a^2}{3r}}$$

Pbm-18

S.T the eqn of the evolute of

$x = a(\cos t + \log\tan t/2)$ and

$y = a\sin t$ is $y = a\cosh x/a$

Soln: $\cosh x = \frac{1}{2}[e^x + e^{-x}]$

Gn Curve $x = a(\cos t + \log\tan \cdot t/2)$

$y = a\sin t$ $\quad \cosh \frac{x}{a} = \frac{1}{2}\{e^{x/a} + e^{-(x/a)}\}$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\sec^2 t/2 \cdot \tfrac{1}{2}\right)$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{\cos t/2}{\sin t/2}\frac{1}{\cos^2 t/2}\frac{1}{2}\right)$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{2\sin t/2 \cos t/2}\right)$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$= a \cot t \cos t$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a\cos t}{a\cot t \cos t}$$

$$= \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t) \frac{dt}{dx}$$

$$= \frac{\sec^2 t}{a\cot t \cos t}$$

$$= \frac{1}{a} \frac{\sin t}{\cos t} \cdot \frac{1}{\cos t} \frac{1}{\cos^2 t}$$

$$= \frac{1}{a} \frac{\sin t}{\cos^4 t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a} \tan t \sec^3 t$$

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$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow x = a(\cos t + \log \tan t/2) - \frac{\tan t(1+\tan^2 t)}{\frac{1}{a}\tan t \sec^3 t}$$

$$\Rightarrow x = a(\cos t + \log \tan t/2) - a\frac{\sec^2 t}{\sec^3 t}$$

$$\Rightarrow x = a(\cos t + \log \tan t/2) - a\frac{1}{\sec t}$$

$$\Rightarrow x = a\cos t + a\log \tan t/2 - a\cos t$$

$$\Rightarrow x = a\log \tan t/2 \longrightarrow (1)$$

Also

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

$$\Rightarrow y = a\sin t + \frac{1+\tan^2 t}{\frac{1}{a}\tan t \sec^3 t}$$

$$\Rightarrow y = a\sin t + a\frac{\sec^2 t}{\tan t \sec^3 t}$$

$$\Rightarrow y = a\sin t + a\frac{1}{\tan t \sec t}$$

$$\Rightarrow y = a\sin t + a\frac{\cos t}{\sin t}\cos t$$

$$\Rightarrow y = a\sin t + \frac{a\cos^2 t}{\sin t}$$

$$= \frac{a\sin^2 t + a\cos^2 t}{\sin t}$$

$$\Rightarrow \quad y = \frac{a}{\sin t} \longrightarrow ②$$

$$① \Rightarrow \quad \frac{x}{a} = \log \tan \frac{t}{2}$$

Taking antilog on b.s, we get

$$e^{x/a} = \tan \frac{t}{2}$$

$$\cosh \frac{x}{a} = \frac{1}{2}\left[e^{x/a} + e^{\left(-x/a\right)}\right]$$

$$\cosh \frac{x}{a} = \frac{1}{2}\left[\tan \frac{t}{2} + \left(\tan \frac{t}{2}\right)^{-1}\right]$$

$$\cosh \frac{x}{a} = \frac{1}{2}\left[\tan \frac{t}{2} + \frac{1}{\tan \frac{t}{2}}\right]$$

$$\cosh \frac{x}{a} = \frac{1}{2}\left[\frac{\tan^2 \frac{t}{2} + 1}{\tan \frac{t}{2}}\right]$$

$$\cosh \frac{x}{a} = \frac{1}{2}\left[\frac{\sec^2 \frac{t}{2}}{\tan \frac{t}{2}}\right]$$

$$\cosh \frac{x}{a} = \frac{1}{2} \quad \frac{1}{\cos^2 \frac{t}{2}} \quad \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}}$$

$$\cosh \frac{x}{a} = \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}$$

$$\cosh \frac{x}{a} = \frac{1}{\sin t}$$

(2) $\Rightarrow y = a \cosh \frac{x}{a}$

The locus of (x, y) is

$y = a \cosh \frac{x}{a}$

Defn :

If the evolute itself be regarded as the original curve, a curve of which it is the evolute is called an involute.

3) H.W $r = a\theta$ Find the p-r eqn and the radius of curvature.

Soln :

Gn Curve : $r = a\theta$

Diff w.r to θ we get.

$$\frac{dr}{d\theta} = a$$

The p-r eqn of the curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}a^2$$

$$\frac{1}{p^2} = \frac{r^2+a^2}{r^4}$$

$$p^2 = \frac{r^4}{r^2+a^2}$$

$$p = \frac{r^2}{(r^2+a^2)^{1/2}}$$

Diff w. r to r we get

$$\frac{dp}{dr} = \frac{(r^2+a^2)^{1/2}(2r) - r^2\frac{1}{2}(r^2+a^2)^{-1/2}(2r)}{\left((r^2+a^2)^{1/2}\right)^2}$$

$$= \frac{2r(r^2+a^2)^{1/2} - r^3\frac{1}{(r^2+a^2)^{1/2}}}{(r^2+a^2)}$$

$$= \frac{\frac{2r(r^2+a^2) - r^3}{(r^2+a^2)^{1/2}}}{(r^2+a^2)}$$

$$= \frac{2r^3 + 2ra^2 - r^3}{(r^2+a^2)^{3/2}}$$

$$= \frac{r^3 + 2ra^2}{(r^2+a^2)^{3/2}}$$

$$\frac{dp}{dr} = \frac{r(r^2+2a^2)}{(r^2+a^2)^{3/2}}$$

$$\frac{1}{r}\frac{dp}{dr} = \frac{\cancel{r}(r^2+2a^2)}{\cancel{r}(r^2+a^2)^{3/2}}$$

$$r\frac{dr}{dp} = \frac{(r^2+a^2)^{3/2}}{(r^2+2a^2)}$$

$$\rho = \frac{(r^2+a^2)^{3/2}}{(r^2+2a^2)}$$

$$\rho = \frac{(a^2\theta^2+a^2)^{3/2}}{(a^2\theta^2+2a^2)}$$

$$\rho = \frac{(a^2(\theta^2+1))^{3/2}}{a^2(\theta^2+2)}$$

$$\rho = \frac{a^3(\theta^2+1)^{3/2}}{a^2(\theta^2+2)}$$

$$\boxed{\rho = \frac{a(\theta^2+1)^{3/2}}{(\theta^2+2)}}$$

Asymtotes

Defn:

If a strought line cuts a curve in two points at infinite distance from the Origin is called an asymtotes to the curve.

To Find the equations of the asymtotes of a plane algebroic curve.

Let the eqn of any curve of the n^{th} degree be arranged in homogeneous sets of terms. Then it can be written as

$$x^n \phi_n (y/x) + x^{n-1} \phi_{n-1} (y/x) + x^{n-2} \phi_{n-2} (y/x) + \dots = 0 \rightarrow (1)$$

Where $\phi_n(y/x)$ is an expression of n^{th} degree in (y/x).

Let us find the straight line cuts the curve.

$y = mx + c$

Putting $y/x = m + c/x$ in (1). we have

$x^n \phi_n (m + c/x) + x^{n-1} \phi_{n-1} (m + c/x) + \dots = 0$

giving the abscissae of the pts of Intersection

Expanding each terms by Taylor's theorem, we have,

$x^n \phi_n (m) + x^{n-1} \left[c\phi_n' (m) + \phi_{n-1} (m) \right] +$

$x^{n-2} \left[\frac{c^2}{2!} \phi_n'' (m) + c.\phi_{n-1}' (m) + \phi_{n-2} (m) \right]$

$+ \dots = 0 \rightarrow (2)$

This is an eqn of the n^{th} degree in x

If $\phi_n (m)$, the co-efficient of the highest power of x be zero, then one root of (2) is infinite of further we equate the co-efficient of x^{n-1} in (2) to zero

$\Rightarrow$ ~~ef0~~

$\Rightarrow c\phi_n' (m) + \phi_{(n-1)} (m) = 0$

In otherwords $y = mx + c$ will be an asymtote if $\phi_n (m) = 0$ & $\rightarrow (i)$

$c \phi_n' (m) + \phi_{n-1} (m) = 0 \rightarrow (ii)$

Since eqn (i) is of n^{th} degree,

there are n values for m (say)

$m_1, m_2, m_3 \dots m_n$

The corresponding values of curve

got from (ii) as

$$c_1 = \frac{-\phi_{n-1}(m_1)}{\phi_n'(m_1)} \quad \&$$

$$c_2 = \frac{-\phi_{n-1}(m_2)}{\phi_n'(m_2)} \quad \dots \text{ etc.}$$

The n asymptotes of a curve (1) are

$$y = m_1 x + c_1$$

$$y = m_2 x + c_2$$

$$y = m_n x + c_n$$

Rule:

In the highest degree terms put $x = 1$ and $y = m$. This gives $\phi_n(m) = 0 \Rightarrow m$ is found, From $\phi_{n-1}(m)$ in a similar manner and differentiate $\phi_{n-1}(m)$. Then the values of c are got from.

$$c = \frac{-\phi_{n-1}(m)}{\phi'_n(m)}, \text{ by putting}$$

$m = m_1, m_2, \ldots m_n$

Remark:

A curve of odd degree cannot have an even number of real asymptotes.

Pbm - 19

Find the asymptotes of the cubic $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$

Soln:

Gn Curve

$$y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$$

The Highest degree terms are

$$y^3 - 6xy^2 + 11x^2y - 6x^3.$$

By rule put $x = 1$ and $y = m$

we get

$$\phi_3 m = m^3 - 6m^2 + 11m - 6 = 0$$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \underline{|0} \\ 2 & 0 & 2 & -6 & \\ \hline & 1 & -3 & \underline{|0} & \\ 3 & 0 & 3 & & \\ \hline & 1 & \underline{|0} & & \end{array}$$

$m = 1, 2, 3$

~~Also~~ Also $\phi_3'(m) = 3m^2 - 12m + 11$

Since ~~here~~ there are no second degree terms, $\phi_2(m) = 0$

$$C = \frac{-\phi_2(m)}{\phi_3'(m)}$$

For $m_1 = 1$, $c_1 = \frac{-0}{3(1) - 12(1) + 11} = \frac{-0}{2} = 0$

For $m_2 = 2$ $c_2 = \frac{0}{3(4) - 12(2) + 11} = \frac{0}{-1} = 0$

For $m_3 = 3$, $c_3 = \frac{0}{3(9) - 12(3) + 11} = \frac{-0}{2} = 0$

The asymptotes are

$y = x$

$y = 2x$

$y = 3x$

-36
11
25.
3

-30
38
2

25
-24

12
11
11

Problem - 20.

Find the asymptotes of

$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$

Soln:

Gn Curve : $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$

The highest degree terms are

$x^3 + 2x^2y - xy^2 - 2y^3$

By rule put $x = 1$ and $y = m$, we get

$\phi_3(m) = 1 + 2m - m^2 - 2m^3 = 0$

$\Rightarrow 2m^3 + m^2 - 2m - 1 = 0$

$\Rightarrow 2m(m^2 - 1) + (m^2 - 1) = 0$

$(m^2 - 1)(2m + 1) = 0$

$m^2 - 1 = 0$ (or) $2m + 1 = 0$

$m^2 = 1$ $\qquad m = \frac{-1}{2}$

$m = \pm 1$

$\therefore m = \pm 1, \frac{-1}{2}$

Also $\phi_3'(m) = 2 - 2m - 6m^2$

$= 2(1 - m - 3m^2)$

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the second degree terms are

$4y^2 + 2xy$

By rule put $x=1$ and $y=m$

$\phi_2(m) = 4m^2 + 2m$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

$$c = \frac{-(4m^2+2m)}{2(1-m-3m^2)}$$

For $m_1 = 1$

$$c_1 = \frac{-(4(1) + 2(1))}{2(1-1-3(1)^2)}$$

$$= \frac{-(4+2)}{2(-3)} = \frac{-6}{-6} = 1$$

$c_1 = 1$

For $m_2 = -1$

$$c_2 = \frac{-(4(-1)^2 + 2(-1))}{2(1-(-1) - 3(-1)^2)}$$

$$= \frac{-(4-2)}{2(2-3)} = \frac{-2}{-2}$$

$c_2 = 1$

For $m_3 = -\frac{1}{2}$

$$c_3 = \frac{-\left(4\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)\right)}{2\left(1-\left(-\frac{1}{2}\right) - 3\left(-\frac{1}{2}\right)^2\right)}$$

$$= \frac{-\left(4\left(\frac{+1}{4}\right) - 1\right)}{2\left(2 + \frac{1}{2} - 3\frac{1}{4}\right)}$$

$$= \frac{-(0)}{2\left(\frac{4+2-3}{4}\right)} = \frac{0}{2\times\frac{3}{4}} = \frac{0}{\frac{3}{2}}$$

$$c_3 = 0$$

The asymptotes are

$y = x + 1$

$y = -x + 1$

$y = -\frac{1}{2}x$

~~Problem~~ :

Special cases:

Let us consider the equations

$\phi_n(m) = 0 \rightarrow ①$ and $c\,\phi'_n(m) + \cancel{\phi_{n+1}(m)}$

$\phi_{n+1}(m) = 0 \rightarrow ②$

Suppose $\phi_n(m) = 0$ has two equal roots (say m_1) then $\phi_n'(m_1) = 0$

If m_1 also satisfies $\phi_{n-1}(m_1) = 0$, then c cannot be determined by ②

In this cas the following term. to determine c,

$$\frac{c^2}{2!}\phi_n''(m) + c_1\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$$

Problem - 21

Find the asymptotes of

$x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y = 1$

Soln:

Given curve

$x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y - 1 = 0$

The highest degree terms are

$x^3 + 2x^2y - 4xy^2 - 8xy^3$

By rule put $x = 1$ and $y = m$, we have

$\phi_3(m) = 1 + 2m - 4m^2 - 8m^3$

$= (1 + 2m) - 4m^2(1 + 2m)$

$= (1 + 2m)(1 - 4m^2)$

$$= (1+2m)(1+2m)(1-2m)$$

$$= (1+2m)^2 (1-2m)$$

Also $\phi_3(m) = 0$

$(1+2m)^2 = 0$ (or) $1-2m = 0$

$$m = -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}$$

$$\phi_3'(m) = 2(1+2m)(2)(1-2m) + (1+2m)^2(-2)$$

$$= 4(1+2m)(1-2m) - 2(1+2m)^2$$

$$= 2(1+2m)\left[2(1-2m) - (1+2m)\right]$$

$$\phi_3'(m) = 2(1+2m)\left[1-6m\right]$$

Since there are no second degree terms, $\phi_2(m) = 0$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

$$= \frac{0}{2(1+2m)(1-6m)}$$

For $m_1 = \frac{-1}{2}$

$$c_1 = \frac{0}{2\left(1+2\left(-\frac{1}{2}\right)\right)\left(1-6\left(-\frac{1}{2}\right)\right)}$$

$$= \frac{0}{2(1-1)(1+3)}$$

$$c_1 = \frac{0}{0}$$

In this case c can't be determined

$\therefore$ Use the formula

$$\frac{c^2}{2}\phi_n''(m) + c\,\phi_{(n-1)}'(m) + \phi_{n-2}(m) = 0 \quad \rightarrow (1)$$

$$\phi_1(m) = -4+8m$$

Now,

$$\phi_3''(m) = 2(1+2m)(-6) + 2(1-6m)(2)$$

$$= 4\left[-3(1+2m) + (1-6m)\right]$$

$$\phi_3''(m) = 4(-2-12m) = -8(1+6m)$$

$$(1) \Rightarrow \frac{c^2}{2}\left[-8(1+6m)\right] + c(0) - 4 + 8m = 0$$

$$\Rightarrow -4c^2(1+6m) - 4 + 8m = 0$$

$$\Rightarrow -4\left(c^2(1+6m) + 1 - 2m\right) = 0$$

$$\Rightarrow c^2(1+6m) + 1 - 2m = 0 \quad \rightarrow (2)$$

For $m_1, m_2 = \frac{-1}{2}$

② $\Rightarrow$ $c^2(1 + 6(\frac{-1}{2})) + 1 - 2(\frac{-1}{2}) = 0$

$c^2(1-3) + 2 = 0$

$-2c^2 + 2 = 0$

$2c^2 = 2$

$c^2 = 1$

$c = \pm 1$

$\boxed{c_1 = 1}$

$\boxed{c_2 = -1}$

For $m_3 = \frac{1}{2}$

② $\Rightarrow$ $c^2(1 + 6(\frac{1}{2})) + 1 - 2(\frac{1}{2}) = 0$

$c^2(1+3) + 1 - 1 = 0$

$4c^2 = 0$

$c^2 = 0$

$\boxed{c_3 = 0}$

The asymptotes are

$$y = \frac{-1}{2}x + 1$$

$$y = -\frac{1}{2}x - 1$$

$$y = \frac{1}{2}x$$

H.W

i) Find the asymptotes of

$$y^2(x^2 - y^2) - 2xy^3 + 2a^3x = 0$$

Soln:

Gn Curve

$$y^2(x^2 - y^2) - 2xy^3 + 2a^3x = 0$$

$$x^2y^2 - y^4 - 2xy^3 + 2a^3x = 0$$

Another Method for finding asymptotes:

Suppose the eqn of the curve of n^{th} degree put in the form.

$(ax + by + c)P_{n-1} + F_{n-1} = 0$, where

P_{n-1} and F_{n-1} denote the polynomials in x and y of $(n-1)^{th}$ degree

Also $ax + by + c = 0$ is called the asymptotic direction and also the asymptote is paralled to $ax+by+c=0$

$\therefore$ The asymptotes for gn curve is

$$(ax+by+c) + \lim_{\substack{y=\frac{-a}{b}x \\ x\to\infty}} \left(\frac{F_{n-1}}{P_{n-1}}\right) = 0$$

Pbm - 22

Find the asymptotes of

$x^3+y^3 = 3axy$

Soln:

Gn Curve : $x^3+y^3 = 3axy$

$\Rightarrow (x+y)(x^2-xy+y^2) - 3axy = 0$

The asympto tes direction is

$x+y=0$

The asymtotes is

$$(x+y) + \lim_{y=-x\to\infty} \frac{-3axy}{x^2-xy+y^2} = 0$$

$$\Rightarrow x+y + \lim_{x\to\infty} \frac{3ax^2}{3x^2} = 0$$

$$x+y + \lim_{x\to\infty} a = 0$$

$\Rightarrow x+y+a=0$ is the required asymptotes.

Pbm - 23.

Find the rectilinear asymptote of $2x^4-5x^2y^2+3y^4+4x^3-6y^3+x^2+y^2-2xy+1=0$

Soln:

Gn Curve

$$2x^4-5x^2y^2+3y^4+4x^3-6y^3+x^2+y^2-2xy+1=0$$

$$\Rightarrow 2x^4-2x^2y^2-3x^2y^2+3y^4+4x^3-6y^3+x^2+y^2-2xy+1=0$$

$$\Rightarrow 2x^2(x^2-y^2)-3y^2(x^2-y^2)+4x^3-6y^3+x^2+y^2-2xy+1=0$$

$$\Rightarrow (2x^2-3y^2)(x^2-y^2)+4x^3-6y^3+x^2+y^2-2xy+1=0$$

$$\Rightarrow (\sqrt{2}x+\sqrt{3}y)(\sqrt{2}x-\sqrt{3}y)(x+y)(x-y)+4x^3-6y^3+x^2+y^2-2xy+1=0$$

The asymptotes direction are,

$(\sqrt{2}x+\sqrt{3}y),(\sqrt{2}x-\sqrt{3}y),(x+y),(x-y)$

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The first asymptotes is

$$(\sqrt{2}x+\sqrt{3}y)+\lim_{y=\frac{-\sqrt{2}}{\sqrt{3}}x\to\infty}\left(\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(\sqrt{2}x-\sqrt{3}y)(x^2-y^2)}\right)=0$$

$$\Rightarrow(\sqrt{2}x+\sqrt{3}y)+\lim_{\sqrt{2}x=-\sqrt{3}y\to\infty}\frac{\sqrt{2}(2x^2)(\sqrt{2}x)-6y^3+x^2+y^2+(\sqrt{2}x)(\sqrt{2}y)+1}{(\sqrt{2}x-\sqrt{3}y)(x^2-y^2)}$$

$$\Rightarrow(\sqrt{2}x+\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{(\sqrt{2})(3y^2)(-\sqrt{3}y)-6y^3+\frac{3}{2}y^2+y^2+(\sqrt{3}y)(\sqrt{2}y)+1}{(-\sqrt{3}y-\sqrt{3}y)(\frac{3}{2}y^2-y^2)}\right)=0$$

$$\Rightarrow(\sqrt{2}x+\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{-3\sqrt{6}y^3-6y^3+\frac{5}{2}y^2+\sqrt{6}y^2+1}{(-2\sqrt{3}y)(\frac{1}{2}y^2)}\right)=0$$

$$\Rightarrow(\sqrt{2}x+\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{-3\sqrt{6}y^3-6y^3+\frac{5}{2}y^2+\sqrt{6}y^2+1}{-\sqrt{3}y^3}\right)=0$$

$$\Rightarrow(\sqrt{2}x+\sqrt{3}y)+\lim_{y\to\infty}\left[\frac{3\sqrt{6}}{\sqrt{3}}+\frac{6}{\sqrt{3}}-\frac{5}{2\sqrt{3}y}-\frac{\sqrt{6}}{\sqrt{3}y}-\frac{1}{\sqrt{3}y^3}\right]=0$$

$$\Rightarrow(\sqrt{2}x+\sqrt{3}y)+3\sqrt{2}+2\sqrt{3}=0$$

The second asymptotes is

$$(\sqrt{2}x-\sqrt{3}y)+\lim_{y=\frac{\sqrt{2}}{\sqrt{3}}x\to\infty}\left(\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(\sqrt{2}x+\sqrt{3}y)(x^2-y^2)}\right)=0$$

$$\Rightarrow(\sqrt{2}x-\sqrt{3}y)+\lim_{\sqrt{2}x=\sqrt{3}y\to\infty}\left(\frac{\sqrt{2}(2x^2)(\sqrt{2}x)-6y^3+x^2+y^2-(\sqrt{2}x)(\sqrt{2}y)+1}{(\sqrt{2}x+\sqrt{3}y)(x^2-y^2)}\right)=0$$

$$\Rightarrow(\sqrt{2}x-\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{\sqrt{2}(3y^2)(\sqrt{3}\sqrt{3})-6y^3+\frac{3}{2}y^2+y^2-(\sqrt{3}y)(\sqrt{2}y)+1}{(\sqrt{3}y+\sqrt{3}y)(\frac{3}{2}y^2-y^2)}\right)=0$$

$$\Rightarrow(\sqrt{2}x-\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{3\sqrt{6}y^3-6y^3+\frac{5}{2}y^2-\sqrt{6}y^2+1}{2\sqrt{3}y\left(\frac{y^2}{2}\right)}\right)=0$$

$$(\sqrt{2}x-\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{3\sqrt{6}y^3-6y^3+\frac{5}{2}y^2-\sqrt{6}y^2+1}{\sqrt{3}y^3}\right)=0$$

$$(\sqrt{2}x-\sqrt{3}y)+\lim_{y\to\infty}\left(\frac{3\sqrt{6}}{\sqrt{3}}-\frac{6}{\sqrt{3}}+\frac{5}{2\sqrt{3}y}-\frac{\sqrt{6}}{\sqrt{3}y}+\frac{1}{\sqrt{3}y^3}\right)=0$$

$$(\sqrt{2}x-\sqrt{3}y)+\left(\frac{3\sqrt{6}}{\sqrt{3}}-\frac{2\sqrt{3}\sqrt{3}}{\sqrt{3}}\right)=0$$

$$(\sqrt{2}x-\sqrt{3}y)+3\sqrt{2}-2\sqrt{3}=0$$

$$(\sqrt{2}x-\sqrt{3}y)+3\sqrt{2}-2\sqrt{3}=0$$

The Third asymptotes is

$$(x+y) + \lim_{y=-x\to\infty}\left(\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(2x^2-3y^2)(x-y)}\right)=0$$

$$(x+y) + \lim_{y\to\infty}\left(\frac{4(-y)^3-6y^3+(-y)^2+y^2-2(-y)(y)+1}{(2(-y)^2-3y^2)(-y-y)}\right)=0$$

$$(x+y) + \lim_{y\to\infty}\left(\frac{-4y^3-6y^3+y^2+y^2+2y^2+1}{(2y^2-3y^2)(-2y)}\right)=0$$

$$(x+y) + \lim_{y\to\infty}\left(\frac{-10y^3+4y^2+1}{(-y^2)(-2y)}\right)=0$$

$$(x+y) + \lim_{y\to\infty}\left(\frac{-10y^3+4y^2+1}{2y^3}\right)=0$$

$$(x+y) + \lim_{y\to\infty}\left(\frac{-10y^3}{2y^3}+\frac{4y^2}{2y^3}+\frac{1}{2y^3}\right)=0$$

$$(x+y) + \lim_{y\to\infty}\left(-5+\frac{2}{y}+\frac{1}{2y^3}\right)=0$$

$$(x+y) + (-5) = 0$$

$$x+y-5=0$$

The Fourth asymptotes is

$$(x-y) + \lim_{y=x\to\infty}\left(\frac{4x^3-6y^3+x^2+y^2-2xy+1}{(2x^2-3y^2)(x+y)}\right)=0$$

$$(x-y) + \lim_{y\to\infty}\left(\frac{4y^3-6y^3+y^2+y^2-2y^2+1}{(2y^2-3y^2)(y+y)}\right)=0$$

$$(x-y) + \lim_{y\to\infty}\left(\frac{-2y^3+1}{(-y^2)(2y)}\right)=0$$

$$(x-y) + \lim_{y\to\infty}\left(\frac{-2y^3+1}{-2y^3}\right)=0$$

$$(x-y) + \lim_{y\to\infty}\left(\frac{-2y^3}{-2y^3}+\frac{1}{-2y^3}\right)=0$$

$$(x-y) + \lim_{y\to\infty}\left(1-\frac{1}{2y^3}\right)=0$$

$$x-y+1=0$$

The asymptotes are

$$(\sqrt{2}x+\sqrt{3}y)+3\sqrt{2}+2\sqrt{3}=0$$

$$(\sqrt{2}x-\sqrt{3}y)+3\sqrt{2}-2\sqrt{3}=0$$

$$x+y-5=0$$

$$x-y+1=0$$

Remark:

1) Suppose the curve is of the form $(ax+by+c)^2 P_{n-2} + F_{n-2} = 0$ then the asymptotes are $(ax+by+c)^2 = \lim_{\substack{y = \frac{-a}{b}x \\ x \to \infty}} \left(\frac{F_{n-2}}{P_{n-2}} \right)$

2) If the curve can be written as $(ax+by)^2 P_{n-2} + (ax+by) F_{n-2} + f_{n-2} = 0$

then the asymptotes are gn by

$$(ax+by)^2 + (ax+by) \lim_{\substack{y = -\frac{a}{b}x \\ x \to \infty}} \left(\frac{F_{n-2}}{P_{n-2}} \right) + \lim_{\substack{y = \frac{-a}{b}x \\ x \to \infty}} \left(\frac{f_{n-2}}{P_{n-2}} \right) = 0$$

Then the parallel asymptotes are $ax+by = \alpha$ and $ax+by = \beta$, where α and β be the roots of the eqn,

$$t^2 + t \lim_{\substack{y = \frac{-a}{b}x \\ x \to \infty}} \left(\frac{F_{n-2}}{P_{n-2}} \right) + \lim_{y = \frac{-a}{b}x} \left(\frac{f_{n-2}}{P_{n-2}} \right) = 0$$

Pbm - 24

Find the asymptotes of

$(x+y)^2 \cdot (x+2y+2) = x + 9y - 2$

Soln:

Gn curve:

$$(x+y)^2 (x+2y+2) = (x+9y-2)$$

The asymptotes parallel to $x+y=0$ is

gn by

$$(x+y)^2 = \lim_{y=-x \to \infty} \left(\frac{x+9y-2}{x+2y+2} \right)$$

$$\Rightarrow (x+y)^2 = \lim_{x \to \infty} \left(\frac{x-9x-2}{x-2x-2} \right)$$

$$\Rightarrow (x+y)^2 = \lim_{x \to \infty} \frac{-8x-2}{-x+2}$$

$$\Rightarrow (x+y)^2 = \lim_{x \to \infty} \left(\frac{x(-8-2/x)}{x(-1+2/x)} \right)$$

$$\Rightarrow (x+y)^2 = \lim_{x \to \infty} \left(\frac{-8-2/x}{-1+2/x} \right)$$

$$\Rightarrow (x+y)^2 = \frac{-8}{-1}$$

$$\Rightarrow (x+y)^2 = 8$$

$$\Rightarrow x+y = \pm 2\sqrt{2}$$

The third asymptotes is parallel to

$x+2y+2=0$

It's eqn. is given by

$$x+2y+2 = \lim_{y=-\frac{x}{2} \to \infty} \left(\frac{x+9y-2}{(x+y)^2} \right)$$

$$\Rightarrow x + 2y + 2 = \lim_{x=-2y\to\infty} \left(\frac{x+9y-2}{(x+y)^2} \right)$$

$$x+2y+2 = \lim_{y\to\infty} \left(\frac{-2y+9y-2}{(-2y+y)^2} \right)$$

$$x+2y+2 = \lim_{y\to\infty} \left(\frac{7y-2}{y^2} \right)$$

$$x+2y+2 = \lim_{y\to\infty} \left(\frac{7}{y} - \frac{2}{y^2} \right)$$

$$x+2y+2=0$$

The asymptotes are

$x+y = \pm\sqrt{2}$ (or)

$x+2y+2=0$

$x+y-\sqrt{2}=0$

$x+y+\sqrt{2}=0$

$x+2y+2=0.$

Pbm - 25

Find the asymptotes of

$(x-y)^2 (x-2y)(x-3y) - 2a(x^3-y^3) - 2a^2(x+y)$

$(x-2y) = 0$

Soln:

Gn curve

$(x-y)^2(x-2y)(x-3y) - 2a(x^3-y^3) - 2a^2(x+y)$

$(x - 2y) = 0$

P_{n-2} F_{n-1}

$(x-y)^2 (x-2y)(x-3y) - 2a(x-y)(x^2+xy+y^2)$

$- 2a^2 (x+y)(x-2y) = 0$

P_{n-2}

The two asymptotes parallel to ~~$x-y$~~.

$x-y=0$ is gn by

$F_{(n-2)}$

$(x-y)^2 + (x-y) \lim_{y=x\to\infty} \left(\frac{-2a(x^2+xy+y^2)}{(x-2y)(x-3y)} \right) +$

P_{n-2}

$\lim_{y=x\to\infty} \left(\frac{-2a^2(x+y)(x-2y)}{(x-2y)(x-3y)} \right) = 0$

P_{n-2}

$\Rightarrow (x-y)^2 - 2a(x-y) \lim_{y=x\to\infty} \left(\frac{x^2+xy+y^2}{(x-2y)(x-3y)} \right) - 2a^2$

$\lim_{y=x\to\infty} \left(\frac{(x+y)}{x-3y} \right) = 0$

$\Rightarrow (x-y)^2 - 2a(x-y) \lim_{x\to\infty} \left(\frac{x^2+x^2+x^2}{(x-2x)(x-3x)} \right)$

$-2a^2 \lim_{x\to\infty} \left(\frac{2x}{x-3x} \right) = 0$

$$\Rightarrow (x-y)^2 - 2a(x-y) \lim_{x\to\infty}\left(\frac{3x^2}{(-x)(-2x)}\right) - 2a^2$$

$$\lim_{x\to\infty}\left(\frac{2x}{-2x}\right) = 0$$

$$\Rightarrow (x-y)^2 - 2a(x-y) \lim_{x\to\infty}\left(\frac{3x^2}{2x^2}\right) - 2a^2$$

$$\lim_{x\to\infty}(-1) = 0$$

$$\Rightarrow (x-y)^2 - 2a(x-y)\,3/2 + 2a^2 = 0$$

$$\Rightarrow (x-y)^2 - 3a(x-y) + 2a^2 = 0$$

$$\Rightarrow ((x-y)-a)((x-y)-2a) = 0$$

$$\Rightarrow x-y-a = 0 \text{ and } x-y-2a = 0$$

→ The third asymptote parallel to $x-2y=0$ is gn by:

$$(x-2y) + \lim_{y=x/2\to\infty}\left(\frac{-2a(x^3-y^3)}{(x-y)^2(x-3y)}\right)$$

$$+ \lim_{y=x/2\to\infty}\left(\frac{-2a^2(x+y)(x-2y)}{(x-y)^2(x-3y)}\right)$$

$$(x-2y) + \lim_{x=2y\to\infty}\left(\frac{-2a(x^3-y^3)}{(x-y)^2(x-3y)}\right)$$

$$+ \lim_{x=2y\to\infty}\left(\frac{-2a^2(x+y)(x-2y)}{(x-y)^2(x-3y)}\right)$$

$$(x-2y) - 2a \lim_{y \to \infty} \left(\frac{(2y)^3 - y^3}{(2y-y)^2(2y-3y)} \right)$$

$$-2a^2 \lim_{y \to \infty} \left(\frac{(2y+y)(2y-2y)}{(2y-y)^2(2y-3y)} \right) = 0$$

$$(x-2y) - 2a \lim_{y \to \infty} \left(\frac{7y^3}{y^2(-y)} \right) - 2a^2$$

$$\lim_{y \to \infty} \left(\frac{0}{y^2(-y)} \right) = 0$$

$$(x-2y) - 2a \lim_{y \to \infty} \left(\frac{-7y^3}{y^3} \right) = 0$$

$$(x-2y) - 2a(-7) = 0$$

$$x - 2y + 14a = 0$$

The forth asymptotes parallel to

$x - 3y = 0$ is gn by.

$$(x-3y) + \lim_{y = x/3 \to \infty} \left(\frac{-2a(x^3 - y^3)}{(x-y)^2(x-2y)} \right) +$$

$$\lim_{y = x/3 \to \infty} \left(\frac{-2a^2(x+y)\cancel{(x-2y)}}{(x-y)^2\cancel{(x-2y)}} \right) = 0$$

$$(x-3y) - 2a \lim_{x = 3y \to \infty} \left(\frac{x^3 - y^3}{(x-y)^2(x-2y)} \right)$$

$$-2a^2 \lim_{x = 3y \to \infty} \left(\frac{x+y}{(x-y)^2} \right) = 0$$

$$(x-3y) - 2a \lim_{y \to \infty} \left(\frac{(3y)^3 - y^3}{(3y-y)^2 \; (3y-2y)} \right)$$

$$-2a^2 \lim_{y \to \infty} \left(\frac{3y+y}{(3y-y)^2} \right) = 0 \qquad (3y-2y)$$

$$(x-3y) - 2a \lim_{y \to \infty} \left(\frac{27y^3 - y^3}{4y^2 \, (y)} \right) - 2a^2$$

$$\lim_{y \to \infty} \left(\frac{4y}{4y^2} \right) = 0$$

$$(x-3y) - 2a \lim_{y \to \infty} \left(\frac{26y^3}{4y^3} \right) - 2a^2 \lim_{y \to \infty} \left(\frac{1}{y} \right) = 0$$

$$(x-3y) - 2a \lim_{y \to \infty} \left(\frac{26}{4} \right) - 2a^2 \lim_{y \to \infty} \left(\frac{1}{y} \right) = 0$$

$$(x-3y) - 2a \left(\frac{26}{4} \right) - 0 = 0$$

$$x - 3y - a(13) = 0$$

$$x - 3y - 13a = 0$$

The asymptotes are

$$x - y - a = 0$$

$$x - y - 2a = 0$$

$$x + 2y + 4a = 0$$

$$x - 3y - 13a = 0$$

H.W

Find the asymptotes of

$y^2(x^2-y^2) - 2ay^3 + 2a^3x = 0.$

Soln:

The given curve

$x^2y^2 - y^4 - 2ay^3 + 2a^3x = 0$

The Highest degree terms are

$x^2y^2 - y^4$

By rule put $x=1$, $y=m$ we get,

$\phi_4(m) = m^2 - m^4 = 0$

$m^2(1-m^2) = 0$

$m^2(1+m)(1-m) = 0$

$m=0$, $m=-1$, $m=1$

$\therefore m = 0, 1, -1$

$\phi_4'(m) = 2m - 4m^3$

The third degree terms are

$-2ay^3$

By rule put $x=1$, $y=m$ we get

$\phi_3(m) = -2am^3$

$$c = \frac{-\phi_3(m)}{\phi_4'(m)} = \frac{-(-2am^3)}{2m - 4m^3}$$

$$c = \frac{2am^3}{2m(1-2m^2)} = \cancel{\frac{2am^2}{2}}$$

$$c = \frac{am^2}{1-2m^2}$$

For $m_1 = 0$

$$c_1 = \frac{a(0)}{1-2(0)} = \frac{0}{1} = 0$$

$$c_1 = 0$$

For $m_2 = 1$

$$c_2 = \frac{a(1)}{1-2(1)} = \frac{a}{-1} = -a$$

$$c_2 = -a$$

For $m_3 = -1$

$$c_3 = \frac{a(-1)}{1-2(1)} = \frac{-a}{-1} = a$$

$$c_3 = a$$

The asymptotes are

$$y = 0$$

$$y = x - a$$

$$y = -x + a$$

Asymptotes by Inspection.

If the eqn of the curve can be put in the form $F_n + F_{n-2} = 0$ where F_n can be break up into linear factors, then $F_n = 0$ represent the required asymptotes.

Pbm - 26.

1m Find the asymptotes of

$$(x+y)(x-y)(x-2y-4) = (3x+7y-6)$$

Soln:

Gn curve:

$$(x+y)(x-y)(x-2y-4) - (3x+7y-6) = 0$$

∴ The gn curve is of the form $F_3 - F_1 = 0$ and F_3 break up into linear factors.

The required asymptotes are

$$(x+y)(x-y)(x-2y-4) = 0$$

$$\Rightarrow x+y = 0, \quad x-y = 0, \quad x-2y-4 = 0$$

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Remark :

1. Any asymptotes of an algebraic curve of degree n cut the curve in n-2 pts.

2) If a curve of degree n has n asymptotes then all intersect the curve in n(n-2) pts.

Unit - IV

Evaluation of Double intergrals

$$\iint f(x,y)\, dx\, dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy\, dx$$

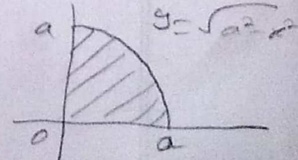
Problem - 1

Evaluate $\iint xy\, dx\, dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$

Soln :

Here x various from 0 to a

y various from 0 to $\sqrt{a^2-x^2}$



$$\iint xy\, dx\, dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x\left(\frac{a^2-x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a x(a^2-x^2)\,dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{1}{2}\left[a^2\int_0^a x\,dx - \int_0^a x^3\,dx\right]$$

$$= \frac{1}{2}\left[a^2\left(\frac{x^2}{2}\right)_0^a - \left(\frac{x^4}{4}\right)_0^a\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2a^4-a^4}{4}\right] = \frac{1}{2}\cdot\frac{a^4}{4}$$

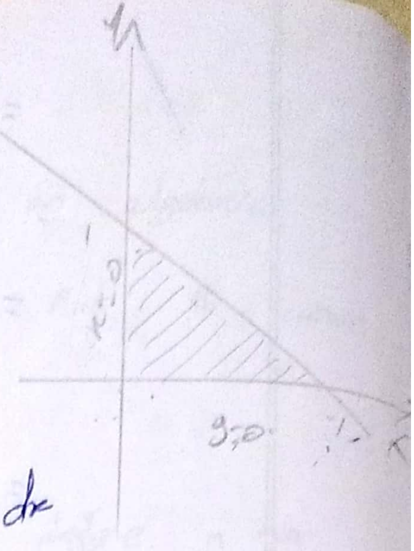
$$= \frac{a^4}{8}$$

Problem - 2

Evaluate $\iint x^2+y^2 \cdot dx\,dy$ over the regoin for which x, y are each ≥ 0 and $x+y \leq 1$

Solution:

The regoin formed by the eqns $x=0$, $y=0$ and $x+y=1$.



$$\iint (x^2+y^2)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left[\int_0^{1-x} \left(x^2 dy + y^2 dy \right) \right] dx$$

$$= \int_0^1 \left[\int_0^{1-x} x^2 dy + \int_0^{1-x} y^2 dy \right] dx$$

$$= \int_0^1 \left[\left(x^2 y \right)_0^{1-x} + \left(\frac{y^3}{3} \right)_0^{1-x} \right] dx$$

$$= \int_0^1 \left[x^2 (1-x) + \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}\left(x - x^3 + 3x^2 - 3x \right) \right] dx$$

$$= \int_0^1 \left[2x^2 - \left(1 + \frac{1}{3}\right) x^3 - x + \frac{1}{3} \right) dx$$

$$= \int_0^1 \left[2x^2 - \frac{4}{3} x^3 - x + \frac{1}{3} \right] dx$$

$$= \left[\frac{2x^3}{3} - \frac{4}{3} \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{3} x \right]_0^1$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{3} - \frac{x^2}{2} + \frac{1}{3}x\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} - \frac{1}{2} + \frac{1}{3}$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{4-3}{6}$$

$$= \frac{1}{6}$$

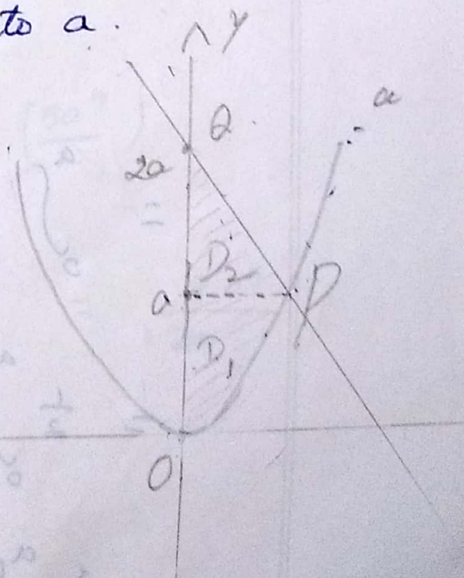
Problem - 3

Change the order of integration integral $\int_0^a \int_{x^2/a}^{2a-x} xy\, dx\, dy$ and evalute it.

Soln:

Here y varies from x^2/a to $2a-x$ and x varies from 0 to a.

Hence the region of integration is OPQ.



This region is divided into two

~~two~~ points say D_1 and D_2

In D_1, x varies from 0 to $\sqrt{ay}$

y varies from 0 to a

In D_2, x varies from 0 to $2a-y$

y varies from a to $2a$

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

$$= \int_0^a \left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} y\,dy + \int_a^{2a} \left(\frac{x^2}{2}\right)_0^{2a-y} y\,dy$$

$$= \int_0^a \frac{ay^2}{2}\,dy + \int_a^{2a} \frac{(2a-y)^2}{2} y\,dy$$

$$= \frac{1}{2}\int_0^a ay^2\,dy + \frac{1}{2}\int_a^{2a} (4a^2+y^2-4ay)y\,dy$$

$$= \frac{1}{2}\int_0^a ay^2\,dy + \frac{1}{2}\int_a^{2a} (4a^2y+y^3-4ay^2)\,dy$$

$$= \frac{1}{2}\left(\frac{ay^3}{3}\right)_0^a + \frac{1}{2}\left(4a^2\frac{y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right)_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(2a^2y^2 + \frac{y^4}{4} - \frac{4ay^3}{3}\right)_a^{2a}$$

$$\cancel{= \frac{a^4}{6} + \frac{1}{2}\left(2a^2y^2 + \frac{y^4}{4} - \frac{5}{3}\right.}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[\left(8a^4 + \frac{16a^4}{4} - \frac{4}{3}(8a^4)\right) - \left(2a^4 + \frac{a^4}{4} - \frac{4}{3}a^4\right)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[6a^4 + 4a^4 - \frac{32}{3}a^4 - \frac{a^4}{4} + \frac{4}{3}a^4\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(10a^4 - \frac{a^4}{3}(32-4) + \frac{a^4}{4}\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(10a^4 - \frac{28}{3}a^4 - \frac{a^4}{4}\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[\frac{120a^4 - 112a^4 - 3a^4}{12}\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{5a^4}{12}\right)$$

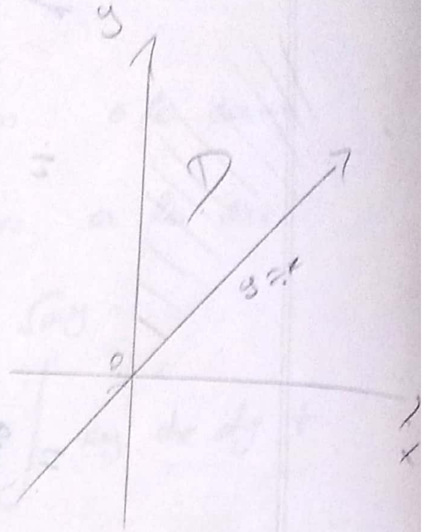
$$= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{9a^4}{24}$$

$$= \frac{3a^4}{8}$$

Problem - 4

By changing the order of Integration evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$

Solution:



In the region D,

y varies from 0 to ∞

and for each fixed y, x varies from 0 to y

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty\int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}(x)_0^y\, dy = \int_0^\infty \frac{e^{-y}}{y}(y)\, dy$$

$$= \int_0^\infty e^{-y}\, dy = \left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= -\left[e^{-\infty} - e^0\right] = -\left[0-1\right]$$

$$= 1$$

Double integral in Polar co-ordinates

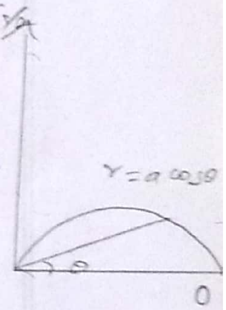
$$\iint_R f(r,\theta)\, r\,dr\,d\theta = \int_{\alpha}\left[\int_{r=f_1(\theta)}^{r=f_2(\theta)} r f(r,\theta)\,dr\right] d\theta$$

Problem-5

Evaluate $\iint r\sqrt{a^2-r^2}\,dr\,d\theta$ over the upper half of the circle $r = a\cos\theta$

Solution:

$$\iint r\sqrt{a^2-r^2}\,dr\,d\theta = \int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$



Put $t = a^2 - r^2$

$dt = -2r\,dr$

$-\frac{dt}{2} = r\,dr$

$$= \int_0^{\pi/2}\int \sqrt{t}\left(\frac{dt}{2}\right) d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\int \sqrt{t}\,dt\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left(\frac{t^{3/2}}{3/2}\right)d\theta = -\frac{1}{2}\times\frac{2}{3}\int_0^{\pi/2} t^{3/2}\,d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left((a^2-r^2)^{3/2}\right)_0^{a\cos\theta} d\theta$$

$$\cancel{= -\frac{1}{3}\int_0^{\pi/2}(a^2 - a^2\cos^2\theta)^{3/2}\,d\theta}$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2 - a^2\cos^2\theta)^{3/2} - (a^2)^{3/2}\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2)^{3/2}(1-\cos^2\theta)^{3/2} - a^3\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[a^3(\sin^2\theta)^{3/2} - a^3\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3)(\sin^3\theta - 1)\, d\theta$$

$$= \frac{-a^3}{3}\left[\int_0^{\pi/2}\sin^3\theta\, d\theta - \int_0^{\pi/2} d\theta\right]$$

$$= \frac{-a^3}{3}\left[\frac{2}{3} - (\theta)_0^{\pi/2}\right]$$

$$= \frac{-a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

$$= \frac{-a^3}{3}\left(\frac{4-3\pi}{6}\right)$$

$$= \frac{(3\pi - 4)\,a^3}{18}$$

Pbm - 6

By transforming into polar co-ordinates evalute $\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy$

over the annular region between the circles $x^2+y^2 = a^2$ and $x^2+y^2 = b^2$ $(b > a)$

Soln:

Put $x = r\cos\theta$ and $y = r\sin\theta$ and $|J| = r$. By transforming into polar-co-ordinates, the two circle becomes $r = a$ and $r = b$.

$$\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$$

$$= \iint_R \frac{r^2\cos^2\theta\; r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= \iint_R \frac{r^5\cos^2\theta\sin^2\theta}{r^2}\,dr\,d\theta$$

$$= \iint_R r^3\cos^2\theta\,\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\,\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^4}{4}\right)_a^b \cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{b^4 - a^4}{4}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta$$

$$= \frac{b^4 - a^4}{4} \int_0^{2\pi} \cos^2\theta (1 - \cos^2\theta)\, d\theta$$

$$= \frac{b^4 - a^4}{4} \int_0^{2\pi} (\cos^2\theta - \cos^4\theta)\, d\theta$$

$$= \frac{b^4 - a^4}{4} \left[\int_0^{2\pi} \cos^2\theta\, d\theta - \int_0^{2\pi} \cos^4\theta\, d\theta \right]$$

$$= \frac{b^4 - a^4}{4} \left[4 \int_0^{\pi/2} \cos^2\theta\, d\theta - 4 \int_0^{\pi/2} \cos^4\theta\, d\theta \right]$$

$$= \frac{4(b^4 - a^4)}{4} \left[\int_0^{\pi/2} \cos^2\theta\, d\theta - \int_0^{\pi/2} \cos^4\theta\, d\theta \right]$$

$$= (b^4 - a^4) \left[\frac{1}{2} \frac{\pi}{2} - \frac{3}{4} \frac{1}{2} \frac{\pi}{2} \right]$$

$$= (b^4 - a^4) \left(\frac{\pi}{4} - \frac{3}{4} \frac{\pi}{4} \right)$$

$$= (b^4 - a^4) \frac{\pi}{4} \left(1 - \frac{3}{4}\right)$$

$$= \frac{(b^4 - a^4)\pi}{4} \left(\frac{1}{4}\right)$$

$$= \left(\frac{b^4 - a^4}{16}\right)\pi$$

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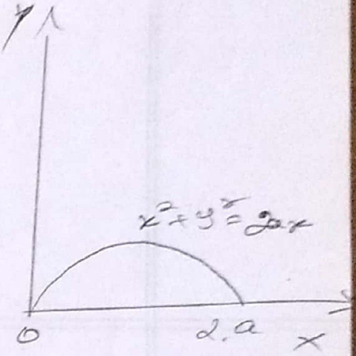
Pbm - 7

By changing into Polar co-ordinats evalute the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$

Soln:

The region of integral is the Seme. circle $x^2+y^2 = 2ax$ above the x axis.



Put $x = r\cos\theta$ $\quad |J| = r$

$y = r\sin\theta$

$x^2+y^2 = 2ax$

$r^2\cos^2\theta + r^2\sin^2\theta = 2ar\cos\theta$

$r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$

r varies from 0 to $2a\cos\theta$

θ varies from 0 to $\pi/2$

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^{2a\cos\theta} (r^2\cos^2\theta + r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{16a^4\cos^4\theta}{4} \, d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= 4a^4 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$= \frac{3\pi a^4}{4}$$

Pbm 8) $\int_0^a \int_0^b (x^2+y^2)\, dx\, dy$

Soln:

$$\int_0^a \int_0^b (x^2+y^2)\, dx\, dy$$

$$= \int_0^a \left[\int_0^b x^2\, dx + y^2 \int_0^b dx\right] dy$$

$$= \int_0^a \left[\left(\frac{x^3}{3}\right)_0^b + y^2 (x)_0^b\right] dy$$

$$= \int_0^a \left(\frac{b^3}{3} + y^2 b\right) dy$$

$$= \frac{b^3}{3}\int_0^a dy + b\int_0^a y^2\, dy$$

$$= \frac{b^3}{3}(y)_0^a + b\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{b^3 a}{3} + \frac{b a^3}{3}$$

$$\int_0^a \int_0^b (x^2+y^2)\, dx\, dy = \frac{ab^3 + a^3 b}{3}$$

Pbm- 9) $\int_0^3 \int_1^2 xy(x+y)\, dy\, dx$

Soln:

$$\int_0^3 \int_1^2 xy(x+y)\, dy\, dx$$

$$= \int_0^3 \int_1^2 (x^2 y + xy^2)\, dy\, dx$$

$$= \int_0^3 \left[x^2\left(\frac{y^2}{2}\right)_1^2 + x\left(\frac{y^3}{3}\right)_1^2\right] dx$$

$$= \int_0^3 \left[x^2 \left[\frac{4}{2} - \frac{1}{2} \right] + x \left[\frac{8}{3} - \frac{1}{3} \right] \right] dx$$

$$= \int_0^3 \left[x^2 \left(\frac{3}{2} \right) + x \left(\frac{7}{3} \right) \right] dx$$

$$= \frac{3}{2} \left(\frac{x^3}{3} \right)_0^3 + \left(\frac{7}{3} \right) \left(\frac{x^2}{2} \right)_0^3$$

$$= \frac{3}{2} \cdot \frac{27}{3} + \frac{7}{3} \left(\frac{9}{2} \right)$$

$$= \frac{27}{2} + \frac{21}{2} = \frac{48}{2}$$

$$= 24$$

Pbm 10) $\int_1^2 \int_1^x xy^2 \, dy \, dx$

Soln:

$\int_1^2 \int_1^x xy^2 \, dy \, dx$

$= \int_1^2 x \left(\frac{y^3}{3}\right)_1^x dx$

$\cancel{= \int_1^2 x \left(\frac{x^3}{3}\right) dx = \frac{1}{3}\int_1^2 x^4 \, dx}$

$\cancel{= \frac{1}{3}\left(\frac{x^5}{5}\right)_1^2 = \frac{1}{3}\left[\frac{32}{5} - \frac{1}{5}\right]}$

$= \int_1^2 x \left[\frac{x^3}{3} - \frac{1}{3}\right] dx$

$= \int_1^2 \left[\frac{x^4}{3} - \frac{x}{3}\right] dx$

$= \frac{1}{3}\left(\frac{x^5}{5}\right)_1^2 - \frac{1}{3}\left(\frac{x^2}{2}\right)_1^2$

$= \frac{1}{3}\left[\frac{32}{5} - \frac{1}{5}\right] - \frac{1}{3}\left[\frac{4}{2} - \frac{1}{2}\right]$

$= \frac{1}{3}\left[\frac{31}{5}\right] - \frac{1}{\cancel{3}}\left[\frac{\cancel{3}}{2}\right]$

$= \frac{62 - 15}{30}$

$\begin{array}{r} 5\,12 \\ 47 \\ 15 \\ \hline 47 \end{array}$

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$$= \frac{47}{30}$$

Pbm-11 $\Rightarrow$ $\int_0^a \int_0^b xy(x-y)\, dy\, dx$

Soln:

$$\int_0^a \int_0^b xy(x-y)\, dy\, dx$$

$$= \int_0^a \int_0^b [x^2y - xy^2]\, dy\, dx$$

$$= \int_0^a \left[x^2\left(\frac{y^2}{2}\right)_0^b - x\left(\frac{y^3}{3}\right)_0^b\right] dx$$

$$= \int_0^a \left[x^2\left(\frac{b^2}{2}\right) - x\left(\frac{b^3}{3}\right)\right] dx$$

$$= \frac{b^2}{2}\left(\frac{x^3}{3}\right)_0^a - \frac{b^3}{3}\left(\frac{x^2}{2}\right)_0^a$$

$$= \frac{b^2}{2}\left(\frac{a^3}{3}\right) - \frac{b^3}{3}\left(\frac{a^2}{2}\right)$$

$$= \frac{a^3b^2}{6} - \frac{a^2b^3}{6}$$

$$= \frac{a^3b^2 - a^2b^3}{6}$$

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Pbm-12) $\int_0^a \int_0^x (x^2+y^2)\, dy\, dx$

Soln:

$$\int_0^a \int_0^x (x^2+y^2)\, dy\, dx$$

$$= \int_0^a \left[x^2 (y)_0^x + \left(\frac{y^3}{3}\right)_0^x\right] dx$$

$$= \int_0^a \left[x^2(x) + \frac{x^3}{3}\right] dx = \int_0^a \left[x^3 + \frac{x^3}{3}\right] dx$$

$$= \left(\frac{x^4}{4}\right)_0^a + \frac{1}{3}\left(\frac{x^4}{4}\right)_0^a$$

$$= \frac{a^4}{4} + \frac{1}{3}\frac{a^4}{4}$$

$$= \frac{a^4}{4} + \frac{a^4}{12}$$

$$= \frac{3a^4+a^4}{12} = \frac{4a^4}{12} = \frac{a^4}{3}$$

Pbm-13) $\int_0^2 \int_{x^2}^{2x} (2x+3y)\, dy\, dx$

Soln

$$\int_0^2 \int_{x^2}^{2x} (2x+3y)\, dy\, dx = \int_0^2 \left[2x(y)_{x^2}^{2x} + 3\left(\frac{y^2}{2}\right)_{x^2}^{2x}\right] dx$$

$$= \int_0^2 \left[2x(2x - x^2) + \frac{3}{2}(4x^2 - x^4)\right] dx$$

$$= \int_0^2 \left[4x^2 - 2x^3 + 6x^2 - \frac{3}{2}x^4\right] dx$$

$$= \int_0^2 \left[10x^2 - 2x^3 - \frac{3}{2}x^4\right] dx$$

$$= 10\left[\frac{x^3}{3}\right]_0^2 - 2\left[\frac{x^4}{4}\right]_0^2 - \frac{3}{2}\left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{10}{3}(8) - \frac{2}{4}(16) - \frac{3}{10}(32)$$

$$= \frac{80}{3} - \frac{32}{4} - \frac{96}{10}$$

$$= \frac{400 - 120 - 144}{15}$$

$$= \frac{400 - 264}{15}$$

$$= \frac{136}{15}$$

Pbm-14) $\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 \, dx \, dy$

Soln:

$$\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 \, dx \, dy$$

$$= \int_0^1 \left(\frac{x^3}{3}\right)_{\sqrt{y}}^{2-y} dy$$

$$= \int_0^1 \left[\frac{(2-y)^3}{3} - \frac{(\sqrt{y})^3}{3}\right] dy$$

$$= \frac{1}{3}\int_0^1 (8 - y^3 - 12y + 6y^2 - y^{3/2})\, dy$$

$$= \frac{1}{3}\left[8y - \frac{y^4}{4} - 12\frac{y^2}{2} + 6\frac{y^3}{3} - \frac{y^{5/2}}{5/2}\right]_0^1$$

$$= \frac{1}{3}\left[8y - \frac{y^4}{4} - 6y^2 + 2y^3 - \frac{2}{5}y^{5/2}\right]_0^1$$

$$= \frac{1}{3}\left[8 - \frac{1}{4} - 6 + 2 - \frac{2}{5}\right]$$

$$= \frac{1}{3}\left[\frac{160 - 5 - 120 + 40 - 8}{20}\right]$$

$$= \frac{200 - 133}{60}$$

$$= \frac{67}{60}$$

Problem - 15

Evalute $I = \int_0^{\pi}\int_r^{a\cos\theta} r\sin\theta \, dr\, d\theta$

Solution :

Gn. $I = \int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr\, d\theta$

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a\cos\theta} \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi} a^2 \cos^2\theta \, \sin\theta \, d\theta$$

$$= \frac{-1}{2} a^2 \int_0^{\pi} \cos^2\theta \, d(\cos\theta)$$

$d(\cos\theta) = -\sin\theta \, d\theta$

$\sin\theta \, d\theta = -d(\cos\theta)$

$$= \frac{-1}{2} a^2 \left(\frac{\cos^3\theta}{3}\right)_0^{\pi}$$

$$= \frac{-1}{6} a^2 (\cos^3\theta)_0^{\pi}$$

$$= \frac{-1}{6} a^2 (\cos^3\pi - \cos^3 0)$$

$\cos 180^\circ = -1$

$$= \frac{-1}{6} a^2 (-1 - 1)$$

$$= \frac{1}{6} a^2 (2)$$

$$I = \frac{1}{3} a^2$$

Pbm - 16

Evalute $\displaystyle\int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2} \, dr \, d\theta$

Soln:

Let $\displaystyle I = \int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2} \, dr \, d\theta$

$r\,dr = \frac{1}{2} d(r^2)$

$$= \int_0^{\pi/2}\left[\int_0^{\infty} \frac{\frac{1}{2} \, d(r^2)}{(r^2+a^2)^2}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{(r^2+a^2)^{-2+1}}{-2+1}\right]_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{-1}{(r^2+a^2)}\right)_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[0-\left(\frac{-1}{a^2}\right)\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{a^2}\right) d\theta$$

$$= \frac{1}{2a^2}(\theta)_0^{\pi/2}$$

$$= \frac{1}{2a^2}\left(\frac{\pi}{2}\right)$$

$$I = \frac{\pi}{4a^2}$$

Pbm - 17

Evaluate $\iint_D x^2y^2\,dx\,dy$, where D is the circular disc $x^2+y^2 \le 1$

Soln:

In D, x varies from -1 to 1

For fixed x, y varies from $-\sqrt{(1-x^2)}$ to $\sqrt{(1-x^2)}$

$$\iint_D x^2y^2\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$$

$$= 4\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$$

$$= 4\int_0^1 x^2\left(\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{4}{3}\int_0^1 x^2\left((1-x^2)^{1/2}\right)^3 dx$$

$$= \frac{4}{3}\int_0^1 x^2(1-x^2)^{3/2}\,dx \longrightarrow (1)$$

Put $x = \sin\theta$

$dx = \cos\theta\,d\theta$

$x = 1 \Rightarrow \theta = \pi/2$

$x = 0 \Rightarrow \theta = 0$

$$(1) \Rightarrow \frac{4}{3}\int_0^{\pi/2} \sin^2\theta\,(\cos^2\theta)^{3/2}\cos\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} (1-\cos^2\theta)\cos^4\theta\,d\theta$$

$$= \frac{4}{3}\left[\int_0^{\pi/2} \cos^4\theta\, d\theta - \int_0^{\pi/2} \cos^6\theta\, d\theta\right]$$

$$= \frac{4}{3}\left[\left(\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right) - \left(\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right)\right]$$

$$= \frac{4}{3}\left[\frac{3\pi}{16} -\right.$$

$$= \frac{4}{3}\left(\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right)\left(1 - \frac{5}{6}\right)$$

$$= \frac{4}{3}\left(\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\,\frac{1}{6}\right)$$

$$= \frac{\pi}{24}$$

Problem- 18

Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$

Soln:

Put $x = r\cos\theta$ and

$y = r\sin\theta$

$\therefore |J| = r$

The region of integration is the entire first quadrant

$\therefore$ r varies from 0 to ∞ &

θ varies from 0 to $\pi/2$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^\infty\int_0^{\pi/2} e^{-r^2}\, r\,d\theta\,dr$$

$$= \int_0^\infty (\theta)_0^{\pi/2} e^{-r^2} r\,dr$$

$$= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\,dr$$

$d(-r^2) =$
$-2r\,dr$
$r\,dr = -\frac{1}{2}d(-r^2)$

$$= \frac{\pi}{2}\int_0^\infty \left(-\frac{1}{2} e^{-r^2} d(-r^2)\right)$$

$$= \frac{-\pi}{4}\int_0^\infty e^{-r^2} d(-r^2)$$

$$= \frac{-\pi}{4}\left(e^{-r^2}\right)_0^\infty = \frac{-\pi}{4}(0-1)$$

$$= \frac{\pi}{4}$$

Pbm - 19

Evalute the integral $\iint r^3 \sin^2 dr\, d\theta$ over the region $r = a\cos\theta$

Soln:

$$\iint r^3 \sin^2 dr\, d\theta = \int_0^{\pi/2}\int_0^{a\cos\theta} r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^{a\cos\theta} \sin^2\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} a^4 \cos^4\theta \sin^2\theta \, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} \cos^4\theta(1-\cos^2\theta)\, d\theta$$

$$= \frac{a^4}{4}\left[\int_0^{\pi/2} \cos^4\theta \, d\theta - \int_0^{\pi/2} \cos^6\theta \, d\theta\right]$$

$$= \frac{a^4}{4}\left[\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) - \left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)\right]$$

$$= \frac{a^4}{4}\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)\left(1-\frac{5}{6}\right)$$

$$= \frac{a^4}{4}\cdot\frac{\cancel{3}^{1}}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{1}{\cancel{6}_2}$$

$$= \frac{\pi a^4}{128}$$

Triple Integrals

$$\int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x,y,z)\, dx\, dy\, dz =$$

$$\int_{z_1}^{z_2}\left[\int_{f_1(z)}^{f_2(z)}\left[\int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x,y,z)\, dx\right] dy\right] dz$$

8m

Pbm - 20

Evaluate $\iiint xyz \, dx \, dy \, dz$ taleous the passitive octant of the sphere. $x^2+y^2+z^2=a^2$

Soln:

Here z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a

$$\therefore \iiint xyz \, dx dy dz$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy \, dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, (a^2-x^2-y^2) \, dy \, dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - y^3x) \, dy \, dx$$

$$= \frac{1}{2} \int_0^a \left(\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{y^4x}{4}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{2a^2xy^2 - 2x^3y^2 - xy^4}{4}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8}\int_0^a \left(2a^2xy^2 - 2x^3y^2 - xy^4\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8}\int_0^a \Big[2a^2x(a^2-x^2) - 2x^3(a^2-x^2) - x(a^2-x^2)^2\Big] dx$$

$$= \frac{1}{8}\int_0^a \Big[2a^4x - 2a^2x^3 - 2a^2x^3 + 2x^5 - x(a^4+x^4-2a^2x^2)\Big] dx$$

$$= \frac{1}{8}\int_0^a \Big[2a^4x - 4a^2x^3 + 2x^5 - a^4x - x^5 + 2a^2x^3\Big] dx$$

$$= \frac{1}{8}\int_0^a \Big[a^4x - 2a^2x^3 + x^5\Big] dx$$

$$= \frac{1}{8}\left[a^4\frac{x^2}{2} - 2a^2\frac{x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[\frac{a^6}{2} - \frac{a^6}{2} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left(\frac{a^6}{6}\right) = \frac{a^6}{48}$$

Pbm - 21

Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

taken over the volume bdd by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$

Soln:

Here z varies from 0 to $1-x-y$

y varies from 0 to $1-x$

x varies from 0 to 1

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3}$$

$$= \int_0^1\int_0^{1-x} \left(\frac{(x+y+z+1)^{-3+1}}{-3+1}\right)_0^{1-x-y} dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x} \left(\frac{1}{(x+y+z+1)^2}\right)_0^{1-x-y} dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x} \left[\frac{1}{(x+y+(1-x-y)+1)^2} - \frac{1}{(x+y+1)^2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right] dy\,dx$$

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$$= \frac{-1}{2}\int_0^1 \left[\frac{1}{4}(y)_0^{1-x} - \left(\frac{(x+y+1)^{-1}}{-1}\right)_0^{1-x}\right] dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1}{4}(1-x) + \left(\frac{1}{(x+y+1)}\right)_0^{1-x}\right] dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1}{4}(1-x) + \left(\frac{1}{x+1-x+1} - \frac{1}{x+1}\right)\right] dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{x+1}\right] dx$$

$$\cancel{= \frac{-1}{2} \quad \frac{1}{4}x - \frac{x^2}{8} + \frac{1}{2}x -}$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right] dx$$

$$= \frac{-1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1)\right]_0^1$$

$$\cancel{= \frac{-1}{2}\left[\frac{3}{4} - \frac{1}{8} - \left[\log(2) + \log(1)\right]\right]}$$

$$= \frac{-1}{2}\left[\left(\frac{3}{4} - \frac{1}{8} - \log(2)\right) + \log(1)\right]$$

$$= \frac{-1}{2}\left(\frac{3}{4} - \frac{1}{8} - \log(2)\right]$$

$$= \frac{-1}{2}\left[\left(\frac{5}{8}\right) - \log(2)\right]$$

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$$= \frac{\not{9}1}{2} \log 2 - \frac{5}{16}$$

Change of ~~of~~ variables.

If $u = f(x, y)$, $v = \phi(x, y)$ be two continuous functions of the independent variables x and y such that

$\frac{\partial y}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are also continuous in x and y.

then
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$
is called the

Jacobian of u and v, w.r to x and y.

and is denoted by $J\left(\frac{u, v}{x, y}\right)$ (or)

$$\frac{\partial (u, v)}{\partial (x, y)}.$$

In the case of these variables u, v, w which are the fuctions of x, y, z. The Jacobian of u, v, w with respect to x, y, z is defined as

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$ and is

denoted by $J\left(\frac{u, v, w}{x, y, z}\right)$ or $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

Theorem - 1

If u, v are functions of x, y and x, y are, themselves functions of ξ, η. Then

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial(u, v)}{\partial(\xi, \eta)}$$

Proof.

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi} & \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} \\ \frac{\partial v}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \xi} & \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \eta} \end{vmatrix}$$

→ ①

But since

$$u = f(x, y) \quad \& \quad v = \phi(x, y)$$

$$x = f_1(\xi, \eta) \quad \& \quad y = f_2(\xi, \eta)$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta}$$

$$① \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{vmatrix} = \frac{\partial(u, v)}{\partial(\xi, \eta)}$$

ξ - ceta

η - Eeta

ζ - Zeta

Theorem - 2

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1$$

Proof:

In the previous result,

put $\xi = u$ and $\eta = v$

we have

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(u,v)}$$

But $\frac{\partial(u,v)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Since u, v are independent variables

$\frac{\partial u}{\partial v} = 0$ $\frac{\partial v}{\partial u} = 0$.

Corollary:

In the case of these variables

i) $\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} = \frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)}$

ii) $\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$

Transformation from cartesian to polar co-ordinates.

Let $x = r\cos\theta$ &

$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Transformation from catesion to spherical polar co-ordinates.

(Imp)

Let $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial z}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(-r^2\sin\theta\cos\phi) + \cancel{r\sin\theta\cos\phi}$$

$$\cancel{(-r\sin^2\theta\sin\phi - r\cos^2\theta}\,.$$

$$= \sin\theta\cos\phi\,(-r^2\sin^2\cos\phi) + r\sin\theta\sin\phi$$

$$(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi) + r\cos\theta\cos\phi\,(-r\sin\theta$$

$$\cos\theta\cos\phi)$$

$$= -r^2\sin^3\theta\cos^2\phi - r^2\sin^3\theta\sin^2\phi$$

$$-r^2\cos^2\theta\sin\theta\sin^2\phi - r^2\sin\theta$$

$$\cos^2\theta\cos^2\phi$$

$$= -r^2\sin^3\theta - r^2\cos^2\theta\sin\theta$$

$$= -r^2\sin\theta\sin^2\theta - r^2\sin\theta\cos^2\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = -r^2\sin\theta$$

Pbm - 22.

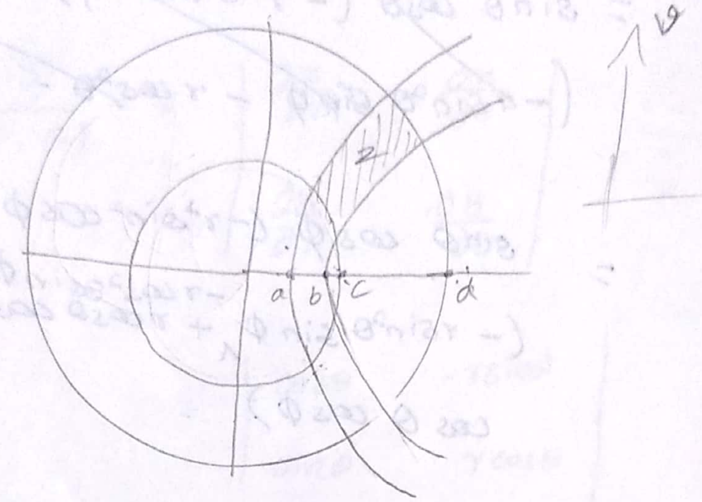
Evaluate $\iint_R xy\,dx\,dy$, where R is the region in the first quadrant. bold by the hyperbolan $x^2-y^2=a^2$ &

$x^2 - y^2 = b^2$ and the circle

$x^2 + y^2 = c^2$ & $x^2 + y^2 = d^2$

$(0 < a < b < c < d)$

Soln:



Put $x^2 - y^2 = u$ &

$x^2 + y^2 = v$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 4xy + 4xy$$

$$= 8xy$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{8xy}$$

In the Rectangle R, $u \to a^2$ to b^2

$v \to c^2$ to d^2

$$\iint xy \, dx \, dy = \int_{a^2}^{b^2} \int_{c^2}^{d^2} xy \frac{1}{8xy} dv \, du.$$

$$= \int_{a^2}^{b^2}\int_{c^2}^{d^2} \frac{1}{8}\, dv\, du$$

$$= \frac{1}{8}\int_{a^2}^{b^2} (v)_{c^2}^{d^2}\, du$$

$$= \frac{1}{8}(d^2-c^2)\int_{a^2}^{b^2} du$$

$$= \frac{(d^2-c^2)}{8}(u)_{a^2}^{b^2}$$

$$= \frac{1}{8}(d^2-c^2)(b^2-a^2)$$

Pbm - 23:

U.Q

Evaluate $\iiint xyz\, dx\, dy\, dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical co-ordinates.

soln:

Let $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$|J| = -r^2\sin\theta$

Here $r \to 0$ to a

$\phi \to 0$ to $\frac{\pi}{2}$

$\theta \to \frac{\pi}{2}$ to 0

$$\iiint xyz\,dx\,dy\,dz$$

$$= \int_0^a \int_0^{\pi/2} \int_{\pi/2}^{0} r^3 \sin^2\theta \cos\theta \cos\phi \sin\phi \,(-r^2 \sin\theta)\, d\theta\, d\phi\, dr$$

$$= \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^5 \sin^3\theta \cos\theta \cos\phi \sin\phi \, d\theta\, d\phi\, dr$$

$$= \int_0^a r^5\,dr \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi$$

$$= \left(\frac{r^6}{6}\right)_0^a \int_0^{\pi/2} \sin^3\theta \, d(\sin\theta) \int_0^{\pi/2} \sin\phi \, d(\sin\phi)$$

$$= \left(\frac{a^6}{6}\right) \left(\frac{\sin^4\theta}{4}\right)_0^{\pi/2} \left(\frac{\sin^2\phi}{2}\right)_0^{\pi/2}$$

$$= \frac{a^6}{6}\,\frac{1}{4}\,\frac{1}{2}\left(\sin^4\tfrac{\pi}{2} - \sin^4 0\right)\left(\sin^2\tfrac{\pi}{2} - \sin^2 0\right)$$

$$= \frac{a^6}{48}(1-0)(1-0)$$

$$= \frac{a^6}{48}$$

Unit - V

Beta and Gamma function:

The Beta function is defined by $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad (m,n>0)$

The Gamma fun is defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \quad (n>0)$$

Remark:

$\Gamma(n)$ cgs (converges) for $n>0$

$\beta(m,n)$ cgs if $m,n>0$

U.Q. (7)

State and Prove: Recurrence formula of Gamma fun.

$$\Gamma(n+1) = n\Gamma(n) \text{ if } n>0$$

Proof:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$$

$$= \lim_{a\to\infty}\left[\int_0^a x^n e^{-x}\,dx\right]$$

$$= \lim_{a\to\infty}\int_0^a x^n\, d(-e^{-x})$$

$$= \lim_{a\to\infty}\left[\left(-x^n e^{-x}\right)_0^a + n\int_0^a e^{-x} x^{n-1}\,dx\right]$$

$$= \lim_{a\to\infty}\left(-x^n e^{-x}\right)_0^a + n\lim_{a\to\infty}\int_0^a e^{-x} x^{n-1}\,dx$$

$u = x^n$

$du = n x^{n-1}\,dx$

$$= \left(-x^n e^{-x}\right)_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\,dx$$

$\int dv = d(-e^{-x})$

$v = -e^{-x}$

$$= (0+0) + n\int_0^\infty e^{-x} x^{n-1}\,dx$$

$$= n\Gamma(n)$$

Properties of Gamma fun

i) $\Gamma(1) = 1$ $\qquad \int_0^\infty e^{-x} x^{n-1}\,dx$

Proof:

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx$$

$$= \left(\frac{e^{-x}}{-1}\right)_0^\infty = \left(-e^{-\infty} + e^{-0}\right)$$

$$= (-0+1) = 1$$

$$\Gamma(1) = 1$$

---

ii) $\Gamma(n+1) = n!$

Proof:

WKT $\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$= n(n-1)(n-2)\cdots 1\,\Gamma(1)$

$= n(n-1)(n-2)\cdots 1$ [by ppty (i)]

$= n!$

$\therefore \Gamma(n+1) = n!$

iii) $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy$

5 mark

Proof: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx.$

Put $x = y^2$

$dx = 2y\,dy$

when $x = 0$, $y = 0$

when $x = \infty$, $y = \infty$

$\therefore \Gamma(n) = \int_0^\infty e^{-y^2} (y^2)^{n-1} (2y\,dy)$

$= 2\int_0^\infty e^{-y^2} y^{2n-2+1}\,dy$

$= 2\int_0^\infty e^{-y^2} y^{2n-1}\,dy$

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Properties of Beta fun.

i) $\beta(m,n) = \beta(n,m)$

Proof:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$$

$(m,n>0)$

Put $x = 1-y \Rightarrow dx = -dy$

when $x=0$, $y=1$

when $x=1$, $y=0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}\,(1-(1-y))^{n-1}\,(-dy)$$

$$\cancel{= \int_1^0 (1-y)^{m-1}}$$

$$= \int_0^1 (1-y)^{m-1}\,y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m).$$

ii) $$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

Proof:

Put $x = \frac{y}{1+y}$

$(1+y)x = y$

$x + xy = y$

$x = y - xy$

$x = (1-x)y$

$\frac{x}{1-x} = y$

when $x=0, \ y=0$

when $x=1, \ y=\infty$

Also $dx = \frac{(1+y)\,dy - y\,dy}{(1+y)^2}$

$dx = \frac{dy}{(1+y)^2}$

$\therefore \beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(1 - \frac{y}{1+y}\right)^{n-1} \left(\frac{dy}{(1+y)^2}\right)$

$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \left(\frac{(1+y-y)}{(1+y)}\right)^{n-1} \frac{dy}{(1+y)^2}$

$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \frac{1}{(1+y)^{n-1}} \frac{dy}{(1+y)^2}$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

iii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\, dx$ (or)

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $x = \sin^2 t$

$dx = 2\sin t \cos t\, dt$

when $x = 0$, $t = 0$

when $x = 1$; $t = \frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2 t)^{m-1}(1-\sin^2 t)^{n-1}\, 2\sin t\cos t\, dt$$

$$= 2\int_0^{\pi/2} \sin^{2m-2} t\, (\cos^2 t)^{n-1} \sin t\cos t\, dt$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1} t\, . \cos^{2n-2+1} t\, dt$$

$$= 2\int_0^{\pi/2} \sin^{2m-1} t \cos^{2n-1} t\, dt$$

Q.B 8m (*) Pbm-1 Named Theorm

Relation between ~~Bet~~ Beta &

Gamma fun.

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Proof:

By ppts (iii) of Γ fun.

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

$$\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\, dy$$

$$\Gamma(m)\,\Gamma(n) = \left(2\int_0^\infty e^{-y^2} y^{2m-1}\, dy\right)\left(2\int_0^\infty e^{-x^2} x^{2n-1}\, dx\right)$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\, dx\, dy$$

Put $x = r\cos\theta$ and $y = r\sin\theta$ and

$|J| = r$

Here $r \to 0$ to ∞ $\quad (x, y \to 0$ to $\infty)$

$\theta \to 0$ to $\pi/2$

$$\therefore \Gamma(m)\,\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1}\, r\, d\theta\, dr$$

$$= 4 \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2n-1+2m-1+1} \cos^{2n-1}\theta \sin^{2m-1}\theta \, d\theta \, dr$$

$$= 4 \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n-1} \cos^{2n-1}\theta \sin^{2m-1}\theta \, d\theta \, dr$$

$$= 4 \int_0^\infty e^{-r^2} r^{2m+2n-1} dr \int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta \sin^{2m-1}\theta \, d\theta$$

$$= 4 \int_0^\infty e^{-r^2} r^{2m+2n-1} dr \left(\frac{1}{2} \beta(m,n) \right)$$

($\because$ By ppty (iii) β fn)

$$= 2 \beta(m,n) \int_0^\infty e^{-r^2} (r^2)^{m+n-1} \frac{1}{2} d(r^2)$$

$$= \beta(m,n) \int_0^\infty e^{-r^2} (r^2)^{m+n-1} d(r^2)$$

$$= \beta(m,n) \Gamma(m+n)$$

$$\Rightarrow \beta(m,n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

Problem - 2

(7) 5 M (or 8 marks) ① P.T $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof:

Put $m = n = \frac{1}{2}$ in the previous result : $\beta(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})}$

$$\Rightarrow \beta(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{(\Gamma(\frac{1}{2}))^2}{\Gamma(1)}$$

$$\Rightarrow \beta(\tfrac{1}{2}, \tfrac{1}{2}) = (\Gamma(\tfrac{1}{2}))^2 \quad [\because \Gamma(1) = 1]$$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = (\beta(\tfrac{1}{2}, \tfrac{1}{2}))^{\frac{1}{2}}$$

$$= \left[2\int_0^{\pi/2} \sin^{2(\frac{1}{2})-1} x \cos^{2(\frac{1}{2})-1} x \, dx\right]^{\frac{1}{2}}$$

$$= \left[2\int_0^{\pi/2} \sin^0 x \cos^0 x \, dx\right]^{\frac{1}{2}}$$

$$= \left[2\int_0^{\pi/2} dx\right]^{\frac{1}{2}} = \left[2\,[x]_0^{\frac{\pi}{2}}\right]^{\frac{1}{2}}$$

$$= \left(2\left(\tfrac{\pi}{2}\right)\right)^{\frac{1}{2}} = (\pi)^{\frac{1}{2}}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

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(4.) Pbm - 3

P.T $\Gamma\left(\frac{P}{2}\right) \Gamma\left(\frac{P+1}{2}\right) = \frac{\sqrt{\pi}}{2^{P-1}} \Gamma(P)$

Proof:

by ppty (iii) of Beta function

$$\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \, dx = \frac{1}{2} \beta(m,n) \rightarrow (1)$$

put $2m = p$ and $2n = q$

$$(1) \Rightarrow \int_0^{\pi/2} \sin^{p-1} x \cos^{q-1} x \, dx = \frac{1}{2} \beta\left(\frac{p}{2}, \frac{q}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(p/2)\Gamma(q/2)}{\Gamma\left(\frac{p+q}{2}\right)} \rightarrow (2) \text{ by pbm-1}$$

Put $q = 1$ in (2); we get

$$\int_0^{\pi/2} \sin^{p-1} x \, dx = \frac{1}{2} \frac{\Gamma(p/2)\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)} \rightarrow (3)$$

Put $p = q$ in (2) we get.

$$\int_0^{\pi/2} \sin^{p-1} x \cos^{p-1} x \, dx = \frac{\frac{1}{2}\Gamma(p/2)\Gamma(p/2)}{\Gamma\left(\frac{p+p}{2}\right)}$$

$$\Rightarrow \int_0^{\pi/2} \sin^{p-1} x \cos^{p-1} x \, dx = \frac{1}{2} \frac{\left(\Gamma(p/2)\right)^2}{\Gamma(p)}$$

$$\Rightarrow \frac{1}{2^{p-1}} \int_0^{\pi/2} 2^{p-1} \sin^{p-1} x \cos^{p-1} x \, dx = \frac{\frac{1}{2}(\Gamma(p/2))^2}{\Gamma(p)}$$

$$\Rightarrow \frac{1}{2^{p-1}} \int_0^{\pi/2} \sin^{p-1} 2x \, dx = \frac{(\Gamma(p/2))^2}{2\,\Gamma(p)}$$

Put $2x = y$

when $x = 0$, $y = 0$

when $x = \pi/2$ $y = \pi$ & $dx = dy/2$

$$\Rightarrow \frac{1}{2^{p-1}} \int_0^{\pi} \sin^{p-1} y \left(\frac{dy}{2}\right) = \frac{(\Gamma(p/2))^2}{2\,\Gamma(p)}$$

$$\Rightarrow \frac{2}{2^{p-1}} \int_0^{\pi/2} \sin^{p-1} y \, dy = \frac{(\Gamma(p/2))^2}{\Gamma(p)}$$

$$\Rightarrow \frac{2}{2^{p-1}} \left[\frac{\frac{1}{2}\Gamma(p/2)\, \Gamma(\frac{1}{2})}{\Gamma(\frac{p+1}{2})} \right] = \frac{(\Gamma(p/2))^2}{\Gamma(p)}$$

[by-⑤]

$$\Rightarrow \frac{1}{2^{p-1}} \frac{\sqrt{\pi}}{\Gamma(\frac{p+1}{2})} = \frac{\Gamma(p/2)}{\Gamma(p)}$$

$$\int_{-1}^{1} f(x)\, dx = \int_0 f(x)\, dx$$

$$\Rightarrow \frac{1}{2^{p-1}} \sqrt{\pi}\, \Gamma(p) = \Gamma(p/2)\, \Gamma\left(\frac{p+1}{2}\right)$$

$$\Rightarrow \Gamma(p/2)\, \Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\, \Gamma(p)$$

Pbm-4

$$P.T \quad \Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

Proof:

Put $p = 2n$ in pbm-3, we have

$$\Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{2n+1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n)$$

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma 2n$$

Problem-5

$$P.T \quad \Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \sqrt{2}\,\pi$$

1 mark

Proof:

Put $p = \frac{1}{2}$ in pbm-3, we have

$$\Gamma(\tfrac{1}{4})\,\Gamma\left(\frac{\frac{1}{2}+1}{2}\right) = \frac{\sqrt{\pi}}{2^{\frac{1}{2}-1}}\,\Gamma(\tfrac{1}{2})$$

$$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \frac{\sqrt{\pi}}{2^{-1/2}}\,\sqrt{\pi}$$

$$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \sqrt{2}\,\pi$$

Pbm-6

Evaluate $\int_0^1 x^m (\log 1/x)^n\,dx$

Soln:

Put $\log \frac{1}{x} = t \Rightarrow \frac{1}{x} = e^t$

$x = e^{-t} \Rightarrow dx = -e^{-t}\,dt$

when $x = 0$, $t = \infty$

when $x = 1$, $t = 0$

$$\int_0^1 x^m \left(\log \tfrac{1}{x}\right)^n dx = \int_\infty^0 (e^{-t})^m (t)^n (-e^{-t}) dt$$

$$= \int_0^\infty e^{-(m+1)t} t^n dt$$

Put $(m+1)t = y$

$$t = \frac{y}{m+1}$$

$$dt = \frac{dy}{m+1}$$

when $t = 0 \Rightarrow y = 0$

when $t = \infty \Rightarrow y = \infty$

$$\int_0^\infty e^{-(m+1)t} t^n dt = \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^n \left(\frac{dy}{m+1}\right)$$

$$= \int_0^\infty \frac{e^{-y} y^n dy}{(m+1)^{n+1}}$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n dy$$

$$= \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

$$= \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

Pbm - 7

Evaluate $\int_0^\infty e^{-x^2}\,dx$

Solution:

Put $x^2 = t$

$2x\,dx = dt$ when $x=0$, $t=0$

$2\sqrt{t}\,dx = dt$ when $x=\infty$, $t=\infty$

$dx = \frac{dt}{2\sqrt{t}}$

$-\frac{1}{2} = n-1$

$n = 1-\frac{1}{2}$

$n = \frac{1}{2}$

$\therefore \int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{t}}$

$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,dt = \frac{1}{2}\int_0^\infty e^{-t}\,t^{\frac{1}{2}-1}\,dt$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$

Problem - 8

Express $\int_0^1 x^m (1-x^n)^p\,dx$ In terms of Gamma function and evaluate the integral $\int_0^1 x^5 (1-x^3)^{10}\,dx$

Solution:

Put $x^n = y \rightarrow (1)$

$x = y^{1/n}$

Diff (1) w.r to x, we get

$$nx^{n-1}\,dx = dy$$

$$dx = \frac{dy}{nx^{n-1}}$$

$$dx = \frac{dy}{n\left(y^{\frac{n-1}{n}}\right)}$$

when $x = 0$ $\quad y = 0$

when $x = 1$ $\quad y = 1$

$$\int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 y^{m/n}(1-y)^p\,\frac{dy}{n\left(y^{\frac{n-1}{n}}\right)}$$

$$= \frac{1}{n}\int_0^1 y^{m/n}\,y^{-\left(\frac{n-1}{n}\right)}(1-y)^p\,dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m-(n-1)}{n}}(1-y)^p\,dy$$

$m-1 = \frac{m-n+1}{n}$

$m = \frac{m-n+1}{n}+1$

$m-1 = p$

$n = p+1$

$$= \frac{1}{n}\,\beta\left(\frac{m-n+1}{n}+1,\ p+1\right)$$

$$= \frac{1}{n}\,\beta\left(\frac{m+1}{n},\ p+1\right)$$

$$= \frac{1}{n}\,\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

$$\therefore \int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\,\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma(10+1)}{\Gamma\left(\frac{5+1}{3}+10+1\right)}$$

$$= \frac{1}{3}\,\frac{\Gamma(2)\,\Gamma(11)}{\Gamma(13)}$$

$$= \frac{1}{3} \frac{1!\ 10!}{12!}$$

$$= \frac{1}{3} \frac{1}{11 \times 12} = \frac{1}{396}$$

Pbm - 9

ST $\Gamma(n+\frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\pi}$

Proof:

$$\Gamma(n+\tfrac{1}{2}) = (n+\tfrac{1}{2}-1)(n+\tfrac{1}{2}-2) \cdots \tfrac{3}{2} \tfrac{1}{2} \Gamma(\tfrac{1}{2})$$

$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2}) \cdots \tfrac{3}{2} \tfrac{1}{2} \sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3) \cdots 3 \cdot 1 \sqrt{\pi}}{2^n}$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \sqrt{\pi}}{2^n}$$

Application of Gamma function to Multiple Integrals

Problem - 10

Evaluate the integral $\iint x^p y^q \, dy \, dx$ over the triangle $x > 0$, $y > 0$, $x + y \leq 1$ in terms of Gamma function.

Solution:

$$\iint x^p y^q \, dy\, dx = \int_0^1 \int_0^{1-x} x^p y^q \, dy \, dx$$

$$= \int_0^1 x^p \left(\frac{y^{q+1}}{q+1} \right)_0^{1-x} dx$$

$$= \frac{1}{q+1} \int_0^1 x^p \left(y^{q+1} \right)_0^{1-x} dx$$

$m-1 = p$

$m = p+1$

$n-1 = q+1$

$n = q+2$

$$= \frac{1}{q+1} \int_0^1 x^p (1-x)^{q+1} dx$$

$$= \frac{1}{q+1} \beta(p+1, q+2)$$

$$= \frac{1}{q+1} \frac{\Gamma(p+1)\,\Gamma(q+2)}{\Gamma(p+q+3)}$$

$$= \frac{1}{q+1} \frac{\Gamma(p+1)(q+1)\,\Gamma(q+1)}{\Gamma(p+q+3)}$$

$$= \frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+3)}$$

i) $\int_0^1 x^7(1-x)^8\,dx$

Solution:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$m = 8$, $n = 9$

$$\int_0^1 x^7(1-x)^8\,dx = \beta(8,9)$$

$$= \frac{\Gamma(8)\ \Gamma(9)}{\Gamma(8+9)} = \frac{\Gamma(8)\Gamma(9)}{\Gamma(17)}$$

$$\int_0^1 x^7(1-x)^8\,dx = \frac{7!\ \ 8!}{16!}$$

ii) $\int_0^{\pi/2} \sin^7\theta\ \cos^5\theta\ d\theta$

Solution:

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta$$

$$\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta$$

$2m-1 = 7$ $\qquad$ $2n-1 = 5$

$m = 4$ $\qquad$ $n = 3$

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$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2}\beta(4,3)$$

$$= \frac{1}{2} \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)}$$

$$= \frac{1}{2} \frac{3!\,2!}{6!}$$

$$= \frac{1}{2} \frac{1\times2\times3\times1\times2}{1\times2\times3\times4\times5\times6}$$

$$= \frac{1}{4\times5\times6}$$

$$= \frac{1}{120}$$

iii) $\int_0^{\pi/2} \sin^{10}\theta \, d\theta$

Solution:

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \int_0^{\pi/2} \sin^{10}\theta \cos\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$m = 10, \quad n = 0$

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{11}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{11}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{11}{2}+\frac{1}{2})}$$

$$= \frac{1}{2} \frac{\Gamma(5+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{12}{2})}$$

$\frac{63\pi}{512}$

$$= \frac{1}{2}\left[\frac{\frac{1.3.5.7.9\sqrt{\pi}}{2^5}\sqrt{\pi}}{5!}\right]$$

$$= \frac{1}{2}\,\frac{1.\cancel{3}.\cancel{5}.7.9\,\pi}{2\times2\times2\times2\times2\times1\times2\times\cancel{3}\times4\times\cancel{5}}$$

$$= \frac{63\pi}{16\times32}$$

$$= \frac{63\pi}{512}$$

Search Pbm - 11

Evaluate the integral $\iint x^p y^q \,dx\,dy$ over the positive quadrant of the circle. $x^2+y^2=a^2$ in therm of Gramma function.

Deduce i) the area of the circle

(ii) The co-ordinates of the ~~center~~ centroid of a quadrant of the circle.

Solution:-

The positive quadrant of the circle is given by the equation

$x \geq 0$, $y \geq 0$, $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 \leq 1$

$\left(\frac{x}{a}\right)^2 = X$

Put $\frac{x}{a} = X^{1/2}$ & $dx = a\,\frac{1}{2}X^{-1/2}\,dX$

$\frac{y}{a} = Y^{1/2}$

$\therefore \iint x^p y^q \,dx\,dy$

$= \iint \left(a(X)^{1/2}\right)^p \left(a(Y)^{1/2}\right)^q \left(a\,\tfrac{1}{2}X^{-1/2}\,dX\right)\left(a\,\tfrac{1}{2}Y^{-1/2}\,dY\right)$

$= \iint \frac{a^{p+q+2}}{4} X^{\frac{p-1}{2}} Y^{\frac{q-1}{2}} \,dX\,dY$

$= \frac{a^{p+q+2}}{4} \iint X^{\frac{p-1}{2}} Y^{\frac{q-1}{2}} \,dX\,dY$

$\cancel{= \frac{a^{p+q+2}}{4} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$

$= \frac{a^{p+q+2}}{4} \int_0^1 \int_0^{1-X} X^{\frac{p-1}{2}} Y^{\frac{q-1}{2}} \,dX\,dY.$

Over the region $X \geq 0$, $Y \geq 0$

$X + Y \leq 1$

$$= \frac{a^{p+q+2}}{4} \int_0^1 x^{\frac{p-1}{2}} \left(\frac{y^{\frac{q-1}{2}+1}}{\frac{q-1}{2}+1} \right)_0^{1-x} dx$$

$$= \frac{a^{p+q+2}}{4} \times \frac{2}{q+1} \int_0^1 x^{\frac{p-1}{2}} \left(y^{\frac{q+1}{2}} \right)_0^{1-x} dx$$

$$= \frac{a^{p+q+2}}{2(q+1)} \int_0^1 x^{\frac{p-1}{2}} (1-x)^{\frac{q+1}{2}} dx$$

$$= \frac{a^{p+q+2}}{2(q+1)} \beta\left(\frac{p-1}{2}+1, \frac{q+1}{2}+1 \right)$$

$$= \frac{a^{p+q+2}}{2(q+1)} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+3}{2}\right)}{\Gamma\left(\frac{p+q}{2}+2\right)}$$

$$= \frac{a^{p+q+2}}{2(q+1)} \frac{\Gamma\left(\frac{p+1}{2}\right) \left(\frac{q+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q}{2}+2\right)}$$

$$= \frac{a^{p+q+2}}{4} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q}{2}+2\right)}$$

i) Area of the circle is $4\iint dx\,dy$

over the region $x \geq 0, y \geq 0$ $x^2+y^2 \leq a^2$

In this case Put $p = 0$ and $q = 0$

$\therefore$ Area of the circle $= 4 \cdot \frac{a^2}{4} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}$

$= \pi a^2$

ii) Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centroid of the quadrant of the circle

$\therefore \bar{x} = \frac{\iint x \, dx dy}{\iint dy \, dx}$, to the integrals being taken over the region

$x \geq 0, y \geq 0, x^2+y^2 \leq a^2$

$|||^{ly}$ $\bar{y} = \frac{\iint y \, dy \, dx}{\iint dx \, dy}$ over the region as mentioned above

To find $\bar{x}$ - Put $p = 1, q = 0$

Numerator of $\bar{x} = \frac{a^3}{4} \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})}$

$= \frac{a^3}{4} \cdot \frac{\sqrt{\pi}}{(\frac{3}{2})\Gamma(\frac{3}{2})}$

$= \frac{a^3}{4} \frac{\sqrt{\pi}}{(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2})}$

$= \frac{a^3}{3} \frac{\sqrt{\pi}}{\sqrt{\pi}}$

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$$= \frac{a^3}{3}$$

$$\therefore \bar{x} = \frac{a^3/3}{\frac{1}{4}(\pi a^2)} = \frac{4a^3}{3\pi a^2} = \frac{4a}{3\pi}$$

To find $\bar{y}$ Put $q = 1$ and $p = 0$

$$\text{Numerator of } \bar{y} = \frac{a^3}{4} \cdot \frac{\Gamma(3/2)\,\Gamma(1)}{\Gamma(5/2)}$$

$$= \frac{a^3}{3}$$

$$\bar{y} = \frac{4a}{3\pi}$$

$$\therefore (\bar{x}, \bar{y}) = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$$

Pbm-12

Evaluate the integral in terms of Gamma fans the integral $\iiint x^p y^q z^r \cdot dx\,dy\,dz$ taken over the volume of tetrahedron gn by

$x \geq 0$, $y \geq 0$, $x+y+z \leq 1$

Solution:

$$\iiint x^p y^q z^r \,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^p y^q z^r \,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} x^p y^q \left(\frac{z^{r+1}}{r+1} \right)_0^{1-x-y} dy\, dx$$

$$= \frac{1}{r+1} \int_0^1 \int_0^{1-x} x^p y^q (1-x-y)^{r+1} dy\, dx$$

Now the region is reduced to

$x \geq 0, \quad y \geq 0, \quad x+y \leq 1$

Put $x+y = u$ and $y = uv$

$x + uv = u$ & $y = uv$

$x = u(1-v)$ & $y = uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

$\therefore dx\, dy = u\, du\, dv$

when $x = 0$, $u(1-v) = 0$

$\Rightarrow u = 0$ (or) $v = 1$

when $y = 0$, $uv = 0$

$u = 0$ (or) $v = 0$

when $x+y = 1$, $u = 1$

Hence the reduced region

becomes $u = 0$, $u = 1$, $v = 0$, $v = 1$ in the

uv plane.

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$\therefore$ The gn integral is

$$= \frac{1}{r+1} \int_0^1 \int_0^1 u^p (1-v)^p u^q v^q (1-u(1-v)-uv)^{r+1} u \, du\, dv$$

$$= \frac{1}{r+1} \int_0^1 \int_0^1 u^{p+q+1} (1-v)^p v^q (1-u)^{r+1} \, du\, dv$$

$$= \frac{1}{r+1} \int_0^1 u^{p+q+1} (1-u)^{r+1} \cdot du \cdot \int_0^1 v^q (1-v)^p \, dv$$

$$= \frac{1}{r+1} \beta(p+q+2, r+2) \, \beta(q+1, p+1)$$

$$= \frac{1}{r+1} \frac{\Gamma(p+q+2)\Gamma(r+2)}{\Gamma(p+q+r+4)} \frac{\Gamma(q+1)\Gamma(p+1)}{\Gamma(p+q+2)}$$

$$= \frac{1}{\cancel{r+1}} \frac{\cancel{(r+1)}\Gamma(r+1)\Gamma(q+1)\Gamma(p+1)}{\Gamma(p+q+r+4)}$$

$$= \frac{\Gamma(r+1)\Gamma(q+1)\Gamma(p+1)}{\Gamma(p+q+r+4)}$$

Pbm - 13

P.T $\iiint \frac{dx\, dy\, dz}{(1-x^2-y^2-z^2)^{1/2}} = \frac{\pi^2}{8}$, the integration extended to all positive values of the variables for which the expression is real.

Soln:

Put $x^2 = X$, $y^2 = Y$, $z^2 = Z$.

$\Rightarrow x = \sqrt{X}$, $y = \sqrt{Y}$, $z = \sqrt{Z}$.

$$\frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \begin{vmatrix} \frac{1}{2}X^{-1/2} & 0 & 0 \\ 0 & \frac{1}{2}Y^{-1/2} & 0 \\ 0 & 0 & \frac{1}{2}Z^{-1/2} \end{vmatrix}$$

$$= \frac{1}{2} X^{-1/2} \left(\frac{1}{4} Y^{-1/2} Z^{-1/2}\right)$$

$$= \frac{1}{8}(XYZ)^{-1/2}$$

$$= \frac{1}{8\sqrt{XYZ}}$$

$$\iiint \frac{dx\, dy\, dz}{(1-x^2-y^2-z^2)^{1/2}} = \frac{1}{8}\int_0^1\int_0^{1-X}\int_0^{1-X-Y} \frac{(1-X-Y-Z)^{-1/2}}{\sqrt{XYZ}}\, dX\, dY\, dZ$$

Over the region $x+y+z \le 1$

Put $z = (1-x-y)\sin^2\theta$

$dz = 2(1-x-y)\sin\theta\cos\theta\, d\theta$

when $z = 0 \Rightarrow \theta = 0$

when $z = 1-x-y \Rightarrow \theta = \frac{\pi}{2}$

$$\frac{1}{8}\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{(1-x-y-z)^{-1/2}}{\sqrt{xyz}}\, dx\, dy\, dz$$

$$= \frac{1}{8}\int_0^1\int_0^{1-x}\int_0^{\pi/2} \frac{(1-x-y-(1-x-y)\sin^2\theta)^{-1/2}}{x^{1/2}\, y^{1/2}\sqrt{1-x-y}\sin\theta}\, 2(1-x-y)\sin\theta\cos\theta\, d\theta\, dy\, dx$$

$$= \frac{1}{4}\int_0^1\int_0^{1-x}\int_0^{\pi/2} \frac{[(1-x-y)(1-\sin^2\theta)]^{-1/2}\sqrt{1-x-y}}{x^{1/2}\, y^{1/2}}\cos\theta\, d\theta\, dy\, dx$$

$$= \frac{1}{4}\int_0^1\int_0^{1-x}\int_0^{\pi/2} \frac{x^{-1/2}\, y^{-1/2}\sqrt{1-x-y}}{\sqrt{1-x-y}\,\sqrt{\cos^2\theta}}\cos\theta\, d\theta\, dy\, dx$$

$$= \frac{1}{4}\int_0^1\int_0^{1-x}\int_0^{\pi/2} x^{-1/2}\, y^{-1/2}\, d\theta\, dy\, dx$$

$$= \frac{1}{4}(\theta)_0^{\pi/2}\int_0^1\int_0^{1-x} x^{-1/2}\, y^{-1/2}\, dy\, dx$$

$$= \frac{\pi}{8} \times \frac{2}{1} \int_0^1 \frac{x^{-1/2}}{(1-x)}$$

$$= \frac{\pi}{8} \int_0^1 \left(\frac{y^{-1/2+1}}{-1/2+1} \right)_0^{1-x} x^{-1/2} \, dx$$

$$= \frac{\pi}{8} \times 2 \int_0^1 x^{-1/2} (1-x)^{1/2} \, dx$$

$$= \frac{\pi}{4} \, \beta\left(-\tfrac{1}{2}+1, \tfrac{1}{2}+1\right)$$

$$= \frac{\pi}{4} \, \beta\left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

$$= \frac{\pi}{4} \, \frac{\Gamma(1/2)\,\Gamma(3/2)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}$$

$$= \frac{\pi}{4} \, \frac{\sqrt{\pi} \; \frac{1}{2}\Gamma(1/2)}{\Gamma(1)}$$

$$= \frac{\pi}{4} \times \frac{1}{2} \sqrt{\pi}\sqrt{\pi} = \frac{\pi^2}{8}$$

1) $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$

Solution:

$$\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}} \, d\theta = \int_0^{\pi/2} \cancel{(\sin\theta)^{1/2}}$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \, \cos^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma(3/4 + 1/4)}$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \frac{\sqrt{2}\,\pi}{1}$$

$$= \frac{\pi}{\sqrt{2}}$$

2) $\int_0^{\pi/2} x^{3/2} (1-x)^{5/2} \, dx$

Solution:

$$\int_0^{\pi/2} x^{3/2} (1-x)^{5/2} \, dx = \cancel{\beta\left(\tfrac{3}{2}, \tfrac{5}{2}\right)}$$

$$= \beta\left(\frac{3}{2}+1, \frac{5}{2}+1\right)$$

$$= \beta\left(\frac{5}{2}, \frac{7}{2}\right)$$

$$= \frac{\Gamma(5/2)\,\Gamma(7/2)}{\Gamma(5/2 + 7/2)}$$

$$= \frac{\frac{3}{2}\Gamma(\frac{3}{2}) \; (\frac{5}{2}) \Gamma(\frac{5}{2})}{\Gamma(6)}$$

$$= \frac{(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2})(\frac{5}{2})(\frac{3}{2})\Gamma(\frac{3}{2})}{5!}$$

$$= \frac{\frac{45}{16}\sqrt{\pi} \; \frac{1}{2}\Gamma(\frac{1}{2})}{1\times2\times3\times4\times5}$$

$$= \frac{\frac{45}{32}\pi}{120}$$

$$= \frac{45\pi}{32\times120}$$

$$= \frac{3\pi}{256}$$

3) P.T $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \pi$

Solution:

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right) \frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right)\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(3/4+1/2)} \quad \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(1/4+1/2)}$$

$$= \frac{1}{4} \frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma\left(\frac{10}{8}\right)} \quad \frac{\Gamma(1/4)\sqrt{\pi}}{\Gamma\left(\frac{6}{8}\right)}$$

$$= \frac{\pi}{4} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(5/4)\Gamma(3/4)}$$

$$= \frac{\pi}{4} \frac{\Gamma(1/4)}{\frac{1}{4}\Gamma(1/4)}$$

$$= \frac{\pi}{4} \times 4$$

$$= \pi$$

4) $$\int_0^{\pi/2} \frac{(\sin^2 x)^{1/3}}{\sqrt{\cos x}}\, dx$$

Solution:

$$\int_0^{\pi/2} \frac{(\sin^2 x)^{1/3}}{\sqrt{\cos x}}\, dx = \int_0^{\pi/2} \sin^{2/3} x \cos^{-1/2} x\, dx$$

$$= \frac{1}{2} \beta\left(\frac{2/3+1}{2}, \frac{-1/2+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(5/6, 1/4\right)$$

$$= \frac{1}{2} \frac{\Gamma(5/6)\,\Gamma(1/4)}{\Gamma\left(\frac{5}{6}+\frac{1}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(5/6)\,\Gamma(1/4)}{\Gamma\left(\frac{26}{24}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(5/6)\,\Gamma(1/4)}{\Gamma(13/12)}$$

$\frac{13}{12}-1$

$1/12$

$$= \frac{1}{2} \frac{\Gamma(5/6)\,\Gamma(1/4)}{(1/12)\,\Gamma(1/12)}$$

$$= \frac{12}{2} \frac{\Gamma(5/6)\,\Gamma(1/4)}{\Gamma(1/12)}$$

$$= 6 \frac{\Gamma(5/6)\,\Gamma(1/4)}{\Gamma(1/12)}$$

Unit - III

Singular Points:

there exists

On some curves, $\exists$ some pts through which more than one branch of the curve pass. Such pts are called Singular Points.

Multiple Points:

If r branches of a curve pass through a point, then the pt is called a multiple points of r^{th} order.

In general, there are r tangents one for each branch at a multiple pt of order.

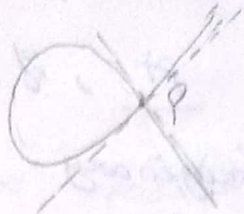
Double Pt:

A pt P is said to be double pt if At that pt P, there must be passes, two branches of a curve

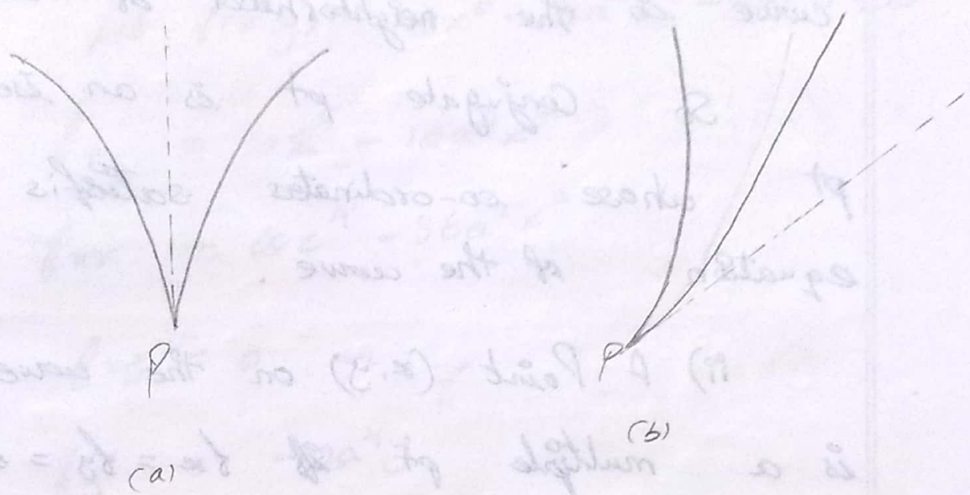
$\parallel^{ly}$ we can define triple pt.

Classification of Double pt:

i) A double pt P is called a node if the tangents at P are real and distinct.



ii) A double pt P is called cusp if the two tangents at P are real and coincident.



a) A cusp P is said to be of first kind if the two branches of the curve lie on the opposite sides of the common tangent at P.

eg) Fig (a)

b) A cusp P is said to be of second kind if the two branches of the curve lie on the same side of the common tangent at P.

(eg). Fig (b)

iii) A double pt P is said to be conjugate pt, if the tangents at P are imaginary.

Remark:

i) If P is the conjugate pt, then there are no real pts on the curve in the neighborhood of that pt.

So Conjugate pt is an isolated pt whose co-ordinates satisfis the equation of the curve.

ii) A Point (x, y) on the curve $f(x,y)=0$ is a multiple pt iff $f_x = f_y = 0$

iii) A double pt is a node if $(f_{xy})^2 - f_{xx} f_{yy} > 0$

iv) A double pt is a cusp if $(f_{xy})^2 - f_{xx} f_{yy} = 0$

v) A double pt is a conjugate pt if $(f_{xy})^2 - f_{xx} f_{yy} < 0$

vi) If $f_{xx} = f_{yy} = f_{xy} = 0$, then the pt (x, y) will be multiple pt of higher order.

Pbm - 1

Find the position and nature of the double pts of the ~~ae~~ curve.

$a^4 y^2 = x^4(2x^2 - 3a^2) \rightarrow (F)$

Soln: Gn curve.

$a^4 y^2 = x^4(2x^2 - 3a^2) \rightarrow (1)$

Let $f(x,y) = 2x^6 - 3a^2x^4 - a^4y^2 = 0$

$f_x = 12x^5 - 12a^2x^3$

$f_{xx} = 60x^4 - 36a^2x^2$

$f_{xy} = 0$

$f_y = -2a^4y$

$f_{yy} = -2a^4$

The double ~~points~~ pts are got from

$f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 12x^5 - 12a^2x^3 = 0$.

$\Rightarrow 12x^3(x^2 - a^2) = 0$

$x^3 = 0$ (or) $x^2 = a^2$

$x = 0$ (or) $x = \pm a$

$f_y = 0 \Rightarrow -2a^4y = 0$

$y = 0$.

Hence the double pts are $(0,0)$, $(a,0)$ $(-a,0)$

In these pts $(0,0)$ only lie on the curve.

$\therefore (0,0)$ is only the double pt.

At $(0,0)$, $f_{xx} = 0$, $f_{yy} = -2a^4$,

$f_{xy} = 0$

$(f_{xy})^2 - f_{xx} f_{yy} = 0^2 - 0(-2a^4)$

$= 0$

The double pt. $(0,0)$ is a cusp

$① \Rightarrow \quad y^2 = \dfrac{x^4(2x^2 - 3a^2)}{a^4}$

$y = \pm \dfrac{x^2}{a^2}(2x^2 - 3a^2)^{\frac{1}{2}}$

Hence the small value of x,

positive (or) negative. $(2x^2 - 3a^2)$ is negative

$\therefore y$ is imaginary

No portion of the curve

lies in the nhod of the origin

Hence the origin is the conjugate

pt not a cusp.

neighbour hood

1) $x^3+x^2+y^2-x-4y+3=0$

Solution :

Given curve

$$x^3+x^2+y^2-x-4y+3=0$$

Let $f(x,y) = x^3+x^2+y^2-x-4y+3=0$

$f_x = 3x^2+2x-1$

$f_{xx} = 6x+2$

$f_{xy} = 0$

$f_y = 2y-4$

$f_{yy} = 2$

The double pts are got from

$f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 3x^2+2x-1=0$

$\Rightarrow 3x^2+3x-x-1=0$

$\Rightarrow 3x(x+1)-1(x+1)=0$

$\Rightarrow (x+1)(3x-1)=0$

$\Rightarrow x=-1$ (or) $x=\frac{1}{3}$

$-1+1+4+1-8+3=0$

$f_y=0 \Rightarrow 2y-4=0$

$\Rightarrow 2y=4$

$\Rightarrow y=2$

$\frac{1}{27}+\frac{1}{9}+4-\frac{1}{3}-8+3$

$\frac{1+3}{27}$

Hence the double pts are $(-1,2)$ $(\frac{1}{3},2)$

In these pts $(-1,2)$ only lie on the

curve.

$\therefore (-1,2)$ is the only double point.

At $(-1,2)$ $f_{xx} = -4$, $f_{yy} = 2$ $f_{xy} = 0$

$(f_{xy})^2 - f_{xx} f_{yy} = (-4)^2 - 2 = 16-2$

$= 14 > 0$

$(f_{xy})^2 - f_{xx} f_{yy} > 0$

The double point $(-1,2)$ is a node.

---

2) $x^4 - 4ax^3 + 2ay^3 + 4a^2x^2 - 3a^2y^2 - a^4 = 0$

Solution:

Given curve

$x^4 - 4ax^3 + 2ay^3 + 4a^2x^2 - 3a^2y^2 - a^4 = 0$

Let $f(x,y) = x^4 - 4ax^3 + 2ay^3 + 4a^2x^2 - 3a^2y^2 - a^4 = 0$

$f_x = 4x^3 - 12ax^2 + 8a^2x$

$f_{xx} = 12x^2 - 24ax + 8a^2$ $\qquad 12a^2 - 24a^2 + 8a^2$

$-4a^2$

$f_{xy} = 0$

$f_y = 6ay^2 - 6a^2y$

$f_{yy} = 12ay - 6a^2$

The double points are got form

$f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 4x^3 - 12ax^2 + 8a^2x = 0$

$4x\left[x^2 - 3ax + 2a^2\right] = 0$

$4x\left[(x-a)(x-2a)\right] = 0$

$x = 0, \quad x = a, \quad x = 2a$

$f_y = 0 \Rightarrow 6ay^2 - 6a^2y = 0$

$6ay\left[y - a\right] = 0$

$y = 0, \quad y = a$

Hence the double points are $(0,0)$ $(a,0)$, $(2a,0)$, $(0,a)$, (a,a), $(2a,a)$.

In these points $(a,0)$ only lie on the curve.

$\therefore$ $(a,0)$ is the only double pt.

At $(a,0)$ $f_{xx} = -4a^2$ $f_{yy} = -6a^2$

$f_{xy} = 0$

$(f_{xy})^2 - f_{xx} f_{yy} = (0)^2 - (-4a^2)(-6a^2)$

$= -24a^4 < 0$

$(f_{xy})^2 - f_{xx} f_{yy} < 0$

The double point $(a,0)$ is a conjugate point.

3) $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$

Solution:

Given Curve:-

$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$

Let $f(x,y) = x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$

$f_x = 4x^3 - 4a^2x$

$f_{xx} = 12x^2 - 4a^2$

$f_{xy} = 0$

$f_y = -6ay^2 - 6a^2y$

$f_{yy} = -12ay - 6a^2$

The double points are got from.

$f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 4x^3 - 4a^2x = 0$

$\Rightarrow 4x(x^2 - a^2) = 0$

$x = 0$ (or) $x = \pm a$

$f_y = 0 \Rightarrow -6ay^2 - 6a^2y = 0$

$\Rightarrow -6ay[y + a] = 0$

$y = 0, \quad y = -a$

Hence the double ~~pad~~ points are

$(0,0)$, $(0,-a)$, $(a,0)$, $(a,-a)$, $(-a,0)$, $(-a,-a)$

In these points $(0,-a)$ $(a,0)$, $(-a,0)$ are lie on the ~~double point~~ curve.

$\therefore$ ~~$a\to$~~ $(0,-a)$, $(a,0)$ $(-a,0)$ are the double ~~b~~ points.

At $(0,-a)$ $f_{xx} = -4a^2$ $f_{yy} = 12a^2 - 6a^2$

$f_{xy} = 0$

$$(f_{xy})^2 - f_{xx} f_{yy} = (0)^2 - (-4a^2)[12a^2 - 6a^2]$$
$$= 48a^4 - 24a^4$$
$$= 24a^2 > 0$$

$(f_{xy})^2 - f_{xx} f_{yy} > 0$

The double point $(0,-a)$ is a node.

At $(a,0)$ $f_{xx} = 8a^2$ $f_{yy} = -6a^2$ $f_{xy} = 0$

$$(f_{xy})^2 - f_{xx} f_{yy} = (0)^2 - (8a^2)(-6a^2)$$
$$= 48a^4 > 0$$

$(f_{xy})^2 - f_{xx} f_{yy} > 0$

The double point $(a,0)$ is a node.

At $(-a,0)$, $f_{xx} = 8a^2$ $f_{yy} = -6a^2$ $f_{xy} = 0$

$$(f_{xy})^2 - f_{xx} f_{yy} = (0)^2 - (8a^2)(-6a^2)$$
$$= 48a^4 > 0$$

$(f_{xy})^2 - f_{xx} f_{yy} > 0$

The double point $(-a,0)$ is a node.

4) $x^2(x-y)+y^2=0$

Solution:

The given curve

$$x^3-x^2y+y^2=0$$

Let $f(x,y)=x^3-x^2y+y^2=0$

$f_x=3x^2-2xy$

$f_{xx}=6x+2y$

$f_{xy}=-2x$

$f_y=-x^2+2y$

$f_{yy}=2$

The double points are got from

$f_x=0$ & $f_y=0$

$f_x=0 \Rightarrow 3x^2-2xy=0$

$x(3x-2y)=0$

$x=0$ (or) $3x-2y=0$

$x=\frac{2y}{3}$

$f_y=0 \Rightarrow -x^2+2y=0$

$2y=x^2$

$y=\frac{x^2}{2}$

$y=0$

$\therefore x=0,\ x=0,\ y=0$

Hence the double points is (0,0)

In this point (0,0) is lie on the curve.

$\therefore$ (0,0) is a double point.

At (0,0) $f_{xx} = 0$, $f_{yy} = 2$ $f_{xy} = 0$

$(f_{xy})^2 - f_{xx} f_{yy} = (0)^2 - (0)(2)$

$= 0$

$(f_{xy})^2 - f_{xx} f_{yy} = 0$

The double point (0,0) is a cusp.

Pbm - 2

Find the position and nature of the double pts of the curve

$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

Solution:

The Given curve

$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

Let $(x,y) = x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

$f_x = 3x^2 + 6xy - 1$

$f_{xx} = 6x + 6y$

$f_{xy} = 6x$

$f_y = 3x^2 - 12y^2 + 1$

$f_{yy} = -24y$

$\frac{1}{27} + 3\left(\frac{1}{9}\right)\left(\frac{1}{3}\right) - 4\left(\frac{1}{27}\right) -$

$\frac{1}{3} + \frac{1}{3} + 3 = 0$

$\frac{1}{27} + \frac{1}{9} - \frac{4}{27} + 3 = 0$

$\frac{27}{81}$

$\frac{27 + 3 - 4 + 81}{27} = 0$

$\frac{108}{27} = 0$ $\frac{30 + 72}{27}$

The double points are got from

$f_x = 0$ & $f_y = 0$

$f_x = 0 \Rightarrow 3x^2 + 6xy - 1 = 0 \Rightarrow 6xy = 1 - 3x^2$

$\cancel{3x(x+2y) - 1 = 0}$ $\quad y = \frac{1-3x^2}{6x}$

$f_y = 0 \Rightarrow 3x^2 - 12y^2 + 1 = 0$

$3x^2 - 12\left(\frac{1-3x^2}{6x}\right)^2 + 1 = 0$

$3x^2 - \frac{12}{36x}(1 + 9x^4 - 6x^2) + 1 = 0$

$3x^2 - \frac{1}{3}(1 + 9x^4 - 6x^2) + 1 = 0$

$\frac{9x^3 - x - 9x^4 + 6x^2 + 3x}{3x} = 0$

$\cancel{9x^2 - x - 9x^4 + 3x = 0}$

$\cancel{9x^4 - 15x^2 + x - 3} = 0$

$\cancel{3x^2(3x^2 - 5) + x - 3} = 0$

$9x^4 - 9x^3 - 6x^2 + x - 3x = 0$

$9x^3(x-1) - 6x^2 - 2x = 0$

$9x^3(x-1)$

$f_y = 0 \Rightarrow 3x^2 - 12y^2 + 1 = 0$

$3x^2 - 12\frac{(1 + 9x^4 - 6x^2)}{36x^2} + 1 = 0$

$3x^2 - \frac{(1 + 9x^4 - 6x^2)}{3x^2} + 1 = 0$

$\frac{9x^4 - 1 - 9x^4 + 6x^2 + 3x^2}{3x^2} = 0$

$$9x^2 - 1 = 0$$

$$9x^2 = 1$$

$$x^2 = \frac{1}{9}$$

$$x = \pm \frac{1}{3}$$

$f_x = 0 \Rightarrow 3\left(\frac{1}{9}\right) + 6\left(\frac{1}{3}\right)y - 1 = 0$

$x = +\frac{1}{3}$ $\qquad \frac{1}{3} + 2y - 1 = 0$

$$2y = 1 - \frac{1}{3} = \frac{2}{3}$$

$$y = \frac{2}{6} = \frac{1}{3}$$

$x = -\frac{1}{3} \Rightarrow 3\left(+\frac{1}{9}\right) + 6\left(-\frac{1}{3}\right)y - 1 = 0$

$$\frac{1}{3} - 2y - 1 = 0$$

$$\frac{1}{3} - 2y - 1 = 0 \Rightarrow -2y = -\frac{1}{3} + 1$$

$$\cancel{-2y = -2} \qquad -2y = \frac{2}{3}$$

$$\cancel{y = 1} \qquad y = -\frac{1}{3}$$

Hence the double points are

$\left(\frac{1}{3}, \frac{1}{3}\right), \left(+\frac{1}{3}, -\frac{1}{3}\right), \left(-\frac{1}{3}, \frac{1}{3}\right), \left(-\frac{1}{3}, -\frac{1}{3}\right)$

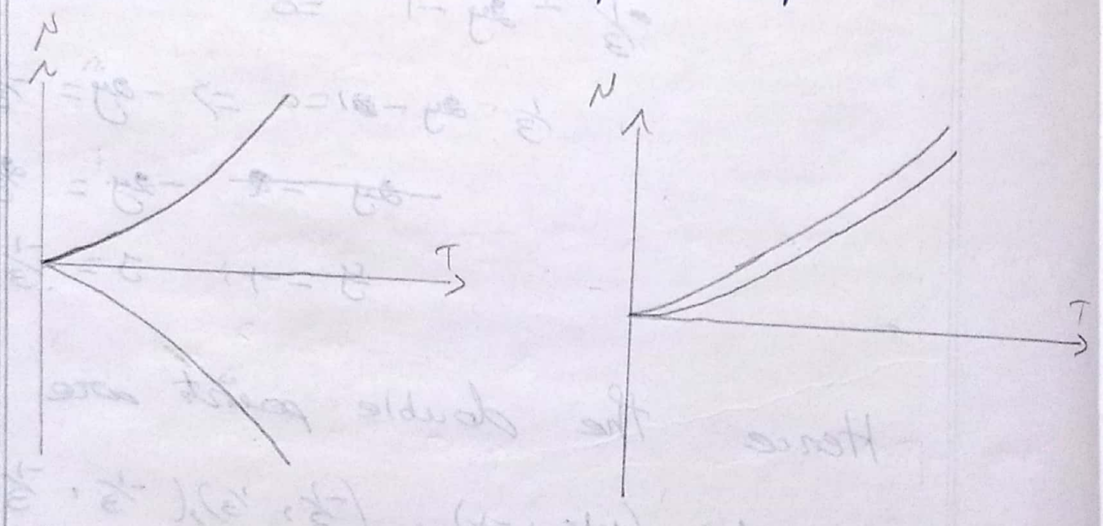
$\therefore$ Hence there are no double points.

Kinds of Cusps:

WKT at a cusp two branches of a curve have a common tangent and hence they have a common normal also.

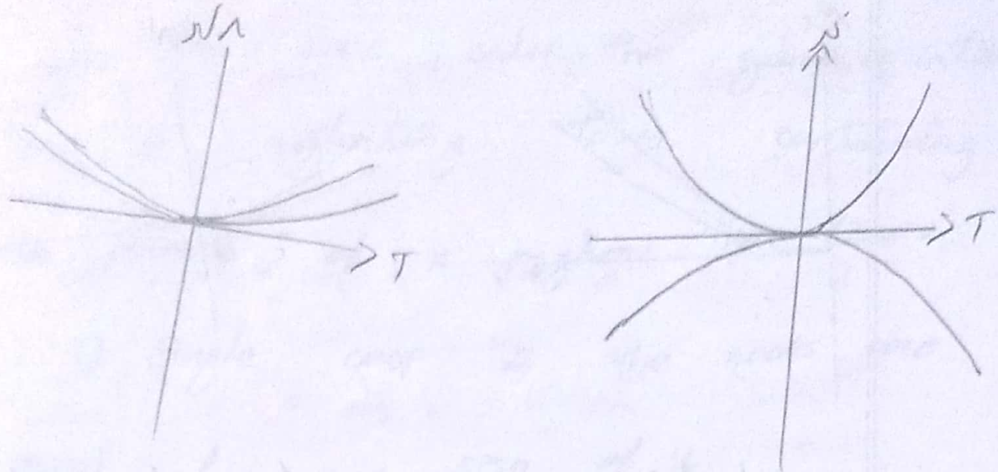
Single cusp:

A cusp is said to be single cusp if the two branches of the curve lie entirely on one side of the common normal at the cusp.



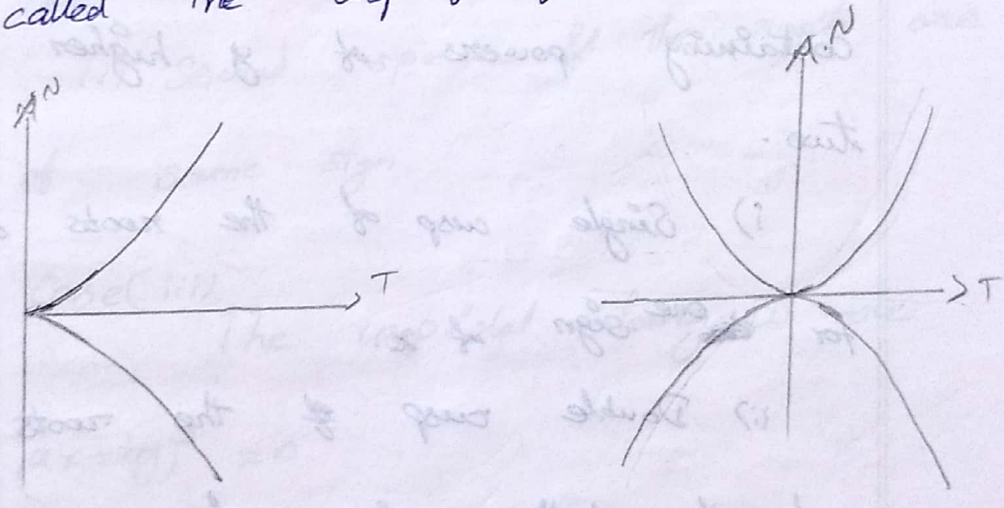
Double cusp:

A cusp is said to be a double cusp if the two branches of the curve extend to both sides of the common normal at cusp.



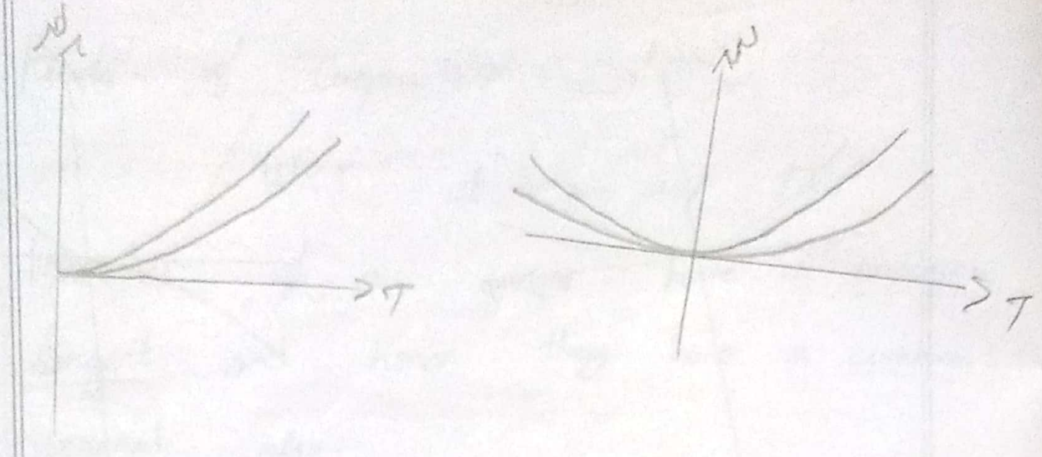
Cusp of first kind: (first species)

If the branches of the curve lie on the opposite sides of the common ~~the~~ tangent at the cusp, the cusp is called the cusp of first kind.



Cusp of second kind (second species)

If the branches of the curve lie on the same side of the common tangent at the cusp, the cusp is called the cusp of second kind.



Working rule to find the nature of the cusp at origin.

Case (i) the cuspidal tangents are $y^2=0$

In this case solve the given equation for y neglective terms containing powers of y higher than two.

i) Single cusp if the roots are real for one sign of x.

ii) Double cusp if the roots are real for both signs of x.

iii) First species if the roots are opposite in sign.

iv) Second species if the roots are of the same sign.

Case (ii) The cuspidal tangents are $x^2=0$

In this case, solve the given equation for x neglecting terms containing the powers of x higher than two.

i) Single cusp if the roots are real for one sign of y

ii) Double cusp if the roots are real for both signs of y.

iii) First species if the roots are opposite in sign.

iv) Second species if the roots are of same sign.

Case (iii)

The cuspidal tangents are

$(ax+by)^2 = 0$

In this case put $p = ax+by$ and eliminate y or x (whichever is convenient) from the given equation of the curve.

Suppose we eliminate y, then we get an equation in p and x.

Solve the equation for p (neglecting p^3 and higher powers of p).

Nature of cusp will be decided as in

case (i) (taking p for y) (or)

case (ii) (taking p for x)

Case (iv) Nature of the cusp at a pt other than the origin.

Transfer the origin to that pt at proceed as in case 1 or case 2 or case 3 may be.

Problem-3

Show that the cusp $y^2(2a-x) = x^3$ → (1) has a single cusp of first species at origin.

Solution:

The given ~~cate~~ curve is of the form

$$x^3 - 2ay^2 + xy^2 = 0 \longrightarrow (1)$$

Equating ~~the~~ to ~~zero~~ zero the lowest degree terms we get

$$-2ay^2 = 0$$

$$\Rightarrow y^2 = 0$$

∴ The roots are real and coincidant

Hence the origin is a cusp (or) conjugate pt.

$x^3 - 2ay^2 + xy^2 = 0$

$x^3 + y^2(x - 2a) = 0$

$y^2(x-2a) = -x^3$

$y^2 = \frac{x^3}{x-2a}$

From ①, we get

$$y^2 = \frac{x^3}{x - 2a}$$

$x > 0$

$2a > 0$

$$y = \pm x \sqrt{\frac{x}{x - 2a}} \longrightarrow ②$$

when x is small and positive, y is real.

Hence the real branches of the curve pass through origin.

∴ The origin is cusp.

Also for any small and positive value of x, the two values of y are of opposite signs.

∴ The cusp is of first species, Also from ②, y is real if x is small and positive.

∴ The cusp is a single cusp.

∴ Origin is a single cusp of first species.

Pbm-4

S.T the curve $y^3 = (x-a)^2 (2x-a)$ has a single cusp of the first species at $(a,0)$.

Soln:

The Equation of the curve is

$$y^3 = (x-a)^2 (2x-a) \longrightarrow (1)$$

Shifting the origin $(a,0)$

by Putting $x = X+a$, $y = Y$

$$(1) \Rightarrow Y^3 = (X+a-a)^2 (2(X+a)-a)$$

$$Y^3 = X^2 (2X+a) \longrightarrow (2)$$

Equating ~~the~~ to zero the lowest degree terms,

we get, $aX^2 = 0$

$X^2 = 0$, whose roots are real and coincident.

Hence the new origin $(a,0)$ is a cusp (or) conjugate pt.

From (2) solving for X, neglecting X^3 and higher powers of X, we get

$$Y^3 = aX^2$$

$$X^2 = Y^3/a$$

$$x = \pm y\sqrt{y/a} \rightarrow (3)$$

When y is ~~s~~small and positive x is ~~neat~~ real.

Hence $(a,0)$ is a cusp.

From (2) for one sign of y, x is real.

$\therefore$ The cusp is single cusp.

Also for any small positive value of y, two values of x are opposite sign. $\therefore$ The cusp is of first species

$\therefore (a,0)$ is a single cusp of first species.

---

1) Find the nature of the cusp $y^2 = x^3$.

Solution.

The given curve is of the form $x^3 - y^2 = 0 \rightarrow (1)$

Equating ~~the~~ to zero the lowest degree terms we get,

$$-y^2 = 0$$

$$y^2 = 0$$

Hence the roots are real and coincident.

~~Hence~~ ∴ The origin is cusp or conjugate pt.

From ①

$$y^2 = x^3$$

$$y = \pm x\sqrt{x} \longrightarrow ②$$

when x is a small and positive, y is real.

Hence the origin is a cusp.

Also any small and positive value of x, the two values of y are opposite signs.

∴ The cusp is first species.

Also from ② y is real if x is small and positive.

∴ The cusp is a single cusp

∴ The origin is a ~~sign~~ single cusp of first species.

2. Find the nature of the cusp.

$y^2 = x^4(x+2)$

Solution:

The give curve is of the form

$y^2 = x^4(x+2) \rightarrow (1)$

$x^5 + 2x^4 - y^2 = 0 \rightarrow (2)$

Equating to zero the lowest degree terms, we het

$-y^2 = 0$

$y^2 = 0$

$\therefore$ The roots are real and coincident.

Hence the origin is a cusp or conjugate pt.

From (1) $y^2 = x^4(x+2)$

$y = \pm x^2\sqrt{x+2}. \longrightarrow (3)$

When x is small ~~and~~ positive and largest ~~lea~~ negative, y is real.

Hence the origin is a cusp.

From (3) for two signs of x,

y is real.

$\therefore$ The cusp is a Double cusp.

Also any small positive and largest negative values of ~~d~~ x, the two values of y are opposite signs

$\therefore$ The cusp is first species.

$\therefore$ The origin is a ~~single~~ Double cusp of first species.

---

3) ~~S.T the curve $y^3 = x^3 + ax^2$ has a single cusp of first species.~~

Curve Tracing.

Suppose a curve is represented interms of Catesian co-ordinates by the equation of $f(x, y) = 0$. The following pts provide the useful informations regarding the the shape and nature of the curve.

I Symmetry of the curve

(a) Symmetry about x-axis

A curve $f(x, y) = 0$ is symmetric about x axis if $f(x, -y) = f(x, y)$

(Eg) $y^2 = 4ax$, $x^2 + y^2 = a^2$

$y^4 + y^2 + x^3 = 0$

$x \to y = -y$
$y \to x = -x$
$(0,0) \to x = -x,\ y = -y$

But $x^2 + y^2 = ay$ is not symmetric about x axis.

(b) Symmetry about y-axis.

A curve $f(x,y) = 0$ is symmetric about y axis if $f(-x,y) = f(x,y)$

(Eg) $x^2 = 4ay$, $x^2 + y^2 = a^2$, $y = x^4 + x^2 + a$

But $x^2 + y^2 = ax$ is not symmetric about y axis.

Note:

$x^2 + y^2 = a^2$ is symmetric about both x and y axes. In this case the equation involves even and only even powers of x and y.

(c) Symmetry about the line $y = x$.

If $f(x,y) = f(y,x)$ then the curve is symmetric about the line $y = x$

(Eg). $x^2 + y^2 = a^2$, $x^3 + y^3 = 3axy$, $xy = c^2$ are symmetric about the line $y=x$.

(d) Symmetry about the origin. (or)

Symmetric in opposite quadrant.

If $f(-x,-y) = f(x,y)$ then the curve is symmetric about the opposite quardrants (or) origin.

(eg)

$x^2+y^2=a^2$, $xy=c^2$ are symmetric about the origin by $x^3+y^3=3axy$, $y^2 = x^3$, are not symmetric ~~about~~ about the origin.

Note:

From the above examples, the equation of the circle ~~curve~~ ~~that~~ has all symmetric pptics.

II Points of intersection with the co-ordinate axes:

To obtain the pts where the curve $f(x,y)=0$ intersects the x axis, put $y=0$ in the gn eqn & solve for x

|||ly to find the pts where the

curve $f(x,y)=0$ intersects the y axis.

Put $x=0$ in the gn equation & solve for y.

(eg) i) The curve $x^2+y^2=a^2$ cuts the x axis at $(a,0)$ & $(-a,0)$, cuts y axis at $(0,a)$ & $(0,-a)$.

ii) The curve $y^2=4ax$ pass through to origin.

III) Region in which the curve lies:

If the equation of the curve $f(x,y)=0$ can be expressed in the form $y=g(x)$, we determine the values of x for which y is imaginary or y is not defined.

Similar information can be obtained if the eqn of the curve can be expressed in the form $x=g(y)$

No portion of the curve lies in the corresponding region.

(eg). The curve $y^2(a-x)=x^3$ can be written as $y=x\sqrt{\frac{x}{a-x}}$ clearly y is imaginary when $x>0$ or $x<a$

Hence the curve does not lie to the left of the y-axis and to the right of the line $x=0$.

IV Tangents to the curve

(a) Tangents at the origin:

If the origin is found to be a point on the curve when the tangents at the origin are obtained by eqn to zero the lowest degree terms occuring in the eqn.

(eg) $y^2 = 4ax$ passes through the origin and the lowest degree term occuring in it is $4ax$ which when equating to zero becomes $4ax = 0$.

(i.e) $x=0$. Hence y-axis is the tangent to the parabola at the $y=0$ are the tangents.

For the curve $a^2y^2 = a^2x^2 - x^4$,

$y = \pm x$ are the tangents at the origin.

(b) Tangents at any other pt (h,k) other than the origin.

Find $\frac{dy}{dx}$ at (h,k) and it gives the slope ~~at~~ of the tangents to the curve at this point. This will be useful to deci-de. the nature of the ~~nature~~ tangents whether parallel to the x-axis or y-axis or inclined tangent.

x Asymptotes:

The concept of ~~+~~ asymptotes described in the previous chapter will be helpful to know about the asymptotes in tracing any curve.

a) Asymptotes parallel to the x-axis.

These are obtained by equating to zero the coefficient of the highest power of x.

(eg) $(y+a)x^2+x-1=0$ has an asymptote $y=-a$ parallel to the x-axis.

(b) Asymptotes parallel to the y axis.

These are obtained by equating to zero the coefficient of the highest power of y.

(eg) $y^2(4-x^2) = x^3 - 1$ has asymptotes

$4 - x^2 = 0$ (i.e) $x = 2$ and $x = -2$ are two asymptotes parallel to the y-axis.

(c) ~~Inclinded~~ Inclined asymptotes.

Taking $y = mx + c$ as an asymptotes we can find m and c by substituting $y = mx + c$ in the equation and equating to zero the various powers of x starting from the highest power.

(eg) ~~For~~ For the curve $x^3 + y^3 = 3axy$,
Put $y = mx + c$
$x + y + a = 0$ is an inclined ~~asymptotes~~ asymptotes.

(vi) Special ~~pot~~ Pts.

Pts at which the function is maximum or minimum ; The pts of inflexion intervals in which the function is increasing or decreasing, region of concavity and convexity, ~~mul~~ ~~multip~~ multiple pts. such as cusp, node, conjugat pts ~~con~~ provide useful informations in determing the shape of the curve.

Having know all these informations by inspecting or investigation we shall trace the curve.

H.W 3) S.T the curve $y^3 = x^3 + ax^2$ ① has a single cusp of first species at the origin.

Solution:

The given curve

$$x^3 - y^3 + ax^2 = 0$$

Equating to zero the lowest degree terms we get

$$ax^2 = 0$$

$$x^2 = 0$$

The root are real and coincident.

Hence the origin is a cusp (or) conjugate pt.

From ①, we get

$$ax^2 = y^3$$

$$x^2 = \frac{y^3}{a}$$

$$x = \pm y\sqrt{\frac{y}{a}} \longrightarrow ②$$

When y is small and positive, x is real.

~~Hence the~~

$\therefore$ The origin is a cusp.

Also any small and positive value of y, the two values of x is ~~at~~ opposite sign

$\therefore$ The cusp is first ~~f~~ species.

From ② x is real if y is small and positive.

$\therefore$ The cusp is ~~sign~~ single cusp.

$\therefore$ The origin is a single cusp of first species.

Pbm-5 Trace the four cusped hypocycloid

(or)

Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$

(Four cusped cycloid or a steroid)

Solution:

The eqn is

$$x^{2/3} + y^{2/3} = a^{2/3} \longrightarrow ①$$

Clearly the curve is symmetrical about both the axes. Hence it is enough to discuss the nature of the curve in the first quadrant only.

To find the points intersection of the curve with x axis, we put

$y=0$ in ①, we get $x^{2/3} = a^{2/3}$

$$\therefore x^2 = a^2$$

hence $x = \pm a$

Hence the curve meets the x axis at $(a,0)$ and $(-a,0)$

Similarly the curve meets the y axis at $(0,a)$ and $(0,-a)$

Rewriting ① as $\left(\frac{y}{a}\right)^{2/3} = 1-\left(\frac{x}{a}\right)^{2/3}$

we see that if $|x| > a$, then $\left(\frac{y}{a}\right)^{2/3} < 0$ and hence y is imaginary.

Hence the curve does not lie beyond $x = \pm a$

Similarly, the curve does not lie beyond $y = \pm a$

Also $\frac{dy}{dx} = -\sqrt[3]{y/x}$

$\therefore \frac{dy}{dx} = 0$ at $(a,0)$ hence x axis is a tangent to the two branches of the curve at $(a,0)$ lying in the first and fourth quadrants.

Hence the curve has a cusp of first kind at $(a,0)$

Similarly the curve has cusps

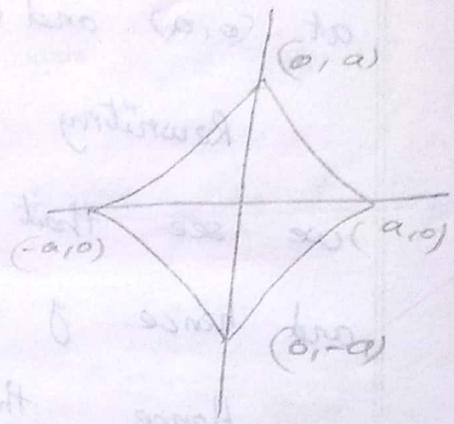
first kind at $(0,a)$ $(-a,0)$ and $(0,-a)$

Hence the curve is know as four cusped hypocycloid.

Also, the curve is concave in $[0,a]$. Hence the form of the curve is as shown in the figure.

~~Note:~~

~~The parametric~~



Note:

The parametric equation of this curve can be taken as $x = a\cos^3\theta$, $y = a\sin^3\theta$

Problem - 6

Trace the curve

$y^2(2a-x) = x^3$ (cissoid)

Solution:

The gn curve

$$y^2(2a-x) = x^3 \longrightarrow (1)$$

Since (1) contains even powers of y the curve is symmetrical about the x-axis obviously it passes through the origin.

The tangents at the origin are given by $y^2 = 0$ and they are real and co-incident.

Hence the origin is a cusp.

The curve meets the x-axis and y-axis only at the origin

Equating the coefficient of the highest degree them is y to zero, we get $x - 2a = 0$

The asymptotes parallel to the y axis is $x - 2a = 0$ and this is the only asymptote of the curve.

Writing the given equation as

$y = x\sqrt{\frac{x}{2a-x}}$ (considering the positive root)

We see that y is imaginary when $x < 0$ (or) $x > 2a$.

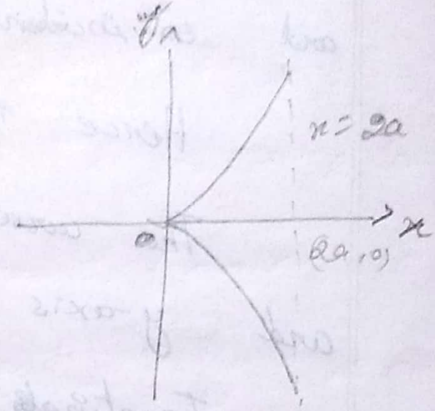
Hence the curve does not lie

to the left of the y-axis and to the right of the line $x = 2a$.

As x increases from 0 to $2a$

y increases from 0 to ∞

Hence the form of the curve is as shown in the figure and the curve is called cissoid.



Pbm - 7

Trace the curve $y^2(a^2+x^2) = x^2(a^2-x^2)$

Soln:

$y^2(2a^2) = a^2(a^2-a^2)$

$y^2 =$

$$y^2(a^2+x^2) = x^2(a^2-x^2) \longrightarrow (1)$$

The power of both x and y are even and hence the curve is symmetrical about ~~the~~ both axes.

The curve obviously passes through the origin.

The tangents at the origin are given the eqn $y^2 = x^2$

Thus the tangents $y = \pm x$ are real and distinct. Hence the origin is a node.

The curve meets the x-axis at $(a,0)$ and $(-a,0)$.

The curve has no asymptotes.

The given eqn can also be written as $y = x\sqrt{\frac{a^2-x^2}{a^2+x^2}}$

$$\therefore \frac{dy}{dx} = \frac{a^4 - 2a^2x^2 - x^4}{(a^2+x^2)^{3/2}(a^2-x^2)^{1/2}}$$

Clearly $\frac{dy}{dx} \to \infty$ as $x \to \pm a$

Hence the tangents to the curve at $(a,0)$ and $(-a,0)$ are parallel to the y-axis.

Now, $\frac{dy}{dx} = 0 \Rightarrow a^4 - 2a^2x^2 - x^4 = 0$

$\div(-1) \Rightarrow x^4 + 2a^2x^2 - a^4 = 0$ $\quad x^2(x^2-2a^2) - a^4 = 0$

$(x^2)^2 - 2a^2x^2 - a^4 = 0$ $\quad x^2 = \frac{-2a^2 \pm \sqrt{8a^4}}{2}$

$a=1, b=-2x^2, c=-a^4$ $\quad x^2 = a^2(-1 \pm \sqrt{2})$

$x^2 =$

The real values of x for which $\frac{dy}{dx} = 0$ are $\pm a\sqrt{(\sqrt{2}-1)}$

~~We~~ Thus the tangents are parallel

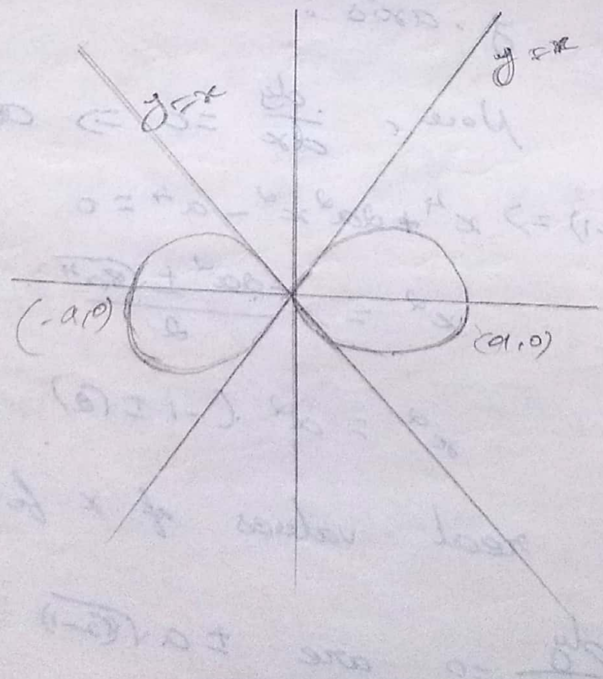
to the x-axis at $x = \pm a\sqrt{(\sqrt{2}-1)}$

We note that y is imaginary if $|x| > a$. Hence the whole ~~ae~~ curve lies between the lines $x = \pm a$.

Obviously the curve passes through the origin. As x increase y also increase and goes on increase until $x = a\sqrt{(\sqrt{2}-1)}$, where $\frac{dy}{dx} = 0$.

(i.e) the tangent is parallel to the x axis. As x increase from $x = a\sqrt{(\sqrt{2}-1)}$ to a, y decreases and finally becomes zero when $x = a$.

The form of the curve is as shown in the figure.



Pbm-8 Trace the curve $x^3+y^3=3axy$

Solution:

$$x^3+y^3=3axy \longrightarrow (1)$$

If x and y are interchanged the equation of the curve (1) is ~~unlo~~ unaltered.

Hence the curve is symmetrical about the line $x=y$.

To find ~~the~~ intersection of the curve with this line we put $x=y$ in (1) we get

$$2x^3=3ax^2$$

$$x^2(2x-3a)=0 \quad \text{Hence } x=0,\ x=\frac{3a}{2}$$

Thus the points of intersection of the curve with the line $x=y$ are $(0,0)$ and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Now, equating the lowest degree terms to zero we get $xy=0$ we get the tangents ~~at~~ $x=0$ and $y=0$ at the Origin. (i.e) The x-axis and y-axis are the tangents at the origin.

The curve has no verti~~cal~~cal asymptotes.

However we can check for the oblique asymptotes by putting $y = mx + c$ in ①, we find m and c by equating to zero the co-efficient of x^3 and x^2 respectively.

We get $x^3 + (mx+c)^3 - 3ax(mx+c) = 0$

$(1+m^3)x^3 + x^2(3m^2c - 3am) + x(3mc^2 - 3ac) + c^3 = 0$

Equating the co-efficient x^3 and x^2 to 0, we get $1+m^2 = 0$ $\quad 3m^2c - 3am = 0$

Now, $1 + m^3 = 0 \Rightarrow m = -1$

$3m^2 c - 3am = c = -a$

Hence $y = -x - a$ is an asymptotes to the curve.

The form of the curve is as shown in the figure.

The curve is known as folium of Descartes.

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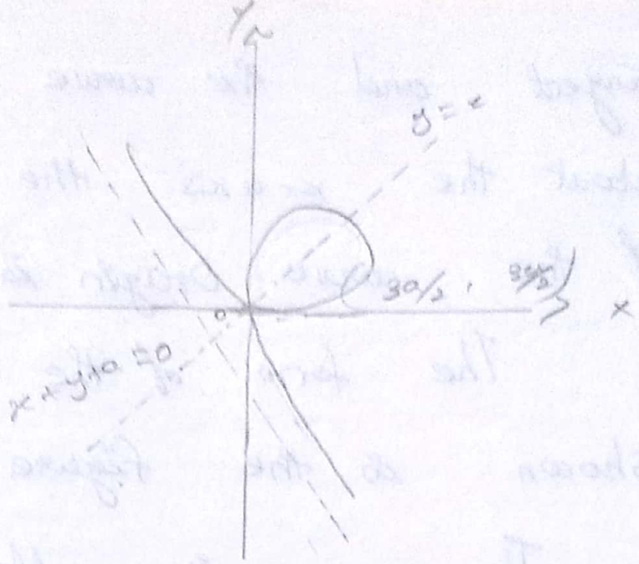

Problem-9

Trace the curve $y^2 = ax^3$

Solution:

The ~~tc~~ curve is symmetric about the x-axis.

It passes through the origin.

It has a tangent $y = 0$ (x-axis) at (0, 0).

The curve has no asymptotes.

Since y is imaginary when $x < 0$, no part of the curve lies to the left of the y-axis.

The curve does not cut the axis except at the origin.

Since the ~~x~~ x axis is the

tangent and the curve is symmetric about the x-axis the two branches of the curve. Origin is a cusp.

The form of the curve is shown in the figure.

The curve is called semi cubical parabola.